Cosmologies spatialement homogènes en théories tenseur-scalaires

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Cosmologies spatialement homogènes en théories tenseur-scalaires

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0.1 Conventions

- Les parenthèses en indice indiquent une opération de symétrisation sur les indices qu'elles renferment.
- Les crochets en indice indiquent une opération d'antisymétrisation sur les indices qu'ils renferment.
- Les symboles de Christoffel seront notés $\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} (g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu})$
- Les composantes du tenseur de Riemann seront notées $R^{\alpha}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}$
- Les composantes du tenseur de Ricci seront notées $R_{\alpha\beta} = R^{\nu}_{\alpha\nu\beta}$
- L'invariant scalaire de courbure sera noté $R = R^{\alpha}_{\alpha}$

Première partie

Introduction

Chapitre 1

Cadre général

Les deux éléments de base servant à définir une cosmologie sont une géométrie représentée mathématiquement par une métrique et un contenu matériel décrit mathématiquement par un Lagrangien. Dans cette thèse, nous allons nous intéresser à une géométrie décrite par les modèles cosmologiques homogènes de Bianchi pour lesquels l'expansion de l'Univers n'est pas la même selon la direction d'observation: elle est donc anisotrope au contraire des modèles classiques de Friedmann-Lemaître-Robertson-Walker (FLRW) où l'expansion est la même quelque soit la direction. En ce qui concerne le contenu matériel, nous considèrerons la présence de champs scalaires dans l'Univers, accompagnés d'un fluide parfait. ***Mais qu'est ce qu'un champ scalaire?*** C'est une fonction qui à chaque point de l'espace et du temps associe un nombre. Un bon exemple de champ scalaire est la température d'une pièce: à chaque point d'une pièce on peut associer une quantité T définissant la température. Un autre exemple est le potentiel gravitationnel ϕ à l'extérieur d'une masse M. Ces champs abondent en physique des particules bien qu'ils n'aient pour l'instant pas encore été détectés et il semble donc normal de les prendre en compte en cosmologie alors que les liens entre ces deux branches de la physique sont de plus en plus étroits. L'intérêt des champs scalaires et des modèles de Bianchi seront discutés en détail dans les chapitres 2 et 3 de cette partie d'un point de vue historique et physique.

Deux questions se posent par rapport à cette description géométrique et physique.
La première question est de savoir pourquoi notre Univers est décrit par un modèle de type FLRW dont la symétrie spatiale est maximale. En effet, bien que cela puisse paraître choquant, il n'y a pas de raison pour que l'expansion soit exactement la même dans toutes les directions ou pour citer R. Feynman à propos de l'idée même de symétrie *"We have, in our minds, a tendancy to accept symmetry as some kind of perfection. In fact it is like the old idea of the Greek that circles were perfect, and it was rather horrible to believe that the planetary orbits were not circles, but only nearly circles.*" Devons nous accepter la perfection des modèles FLRW ou devons nous l'abandonner au profit d'un Univers approximativement parfait? A cette question correspond essentiellement deux courants d'idées: l'un postule l'existence d'un principe quantique comme une théorie des conditions initiales qui sélectionnerait parmi l'ensemble des modèles possibles les modèles de type FLRW, l'autre l'existence d'un Univers primordial moins symétrique qu'un modèle FLRW mais evoluant dynamiquement vers ce dernier. C'est ce dernier point de vue que nous adopterons ici.
La seconde question concerne les propriétés des champs scalaires théoriquement présents dans notre Univers. En effet, il existe une infinité de théories tenseur-scalaires possibles. Aussi, il est nécessaire d'être capable d'éliminer celles conduisant à des résultats physique abérants ou au contraire de repérer celles menant à des comportements physiquement intéressants pour notre Univers. C'est à cette deuxième question que nous allons tenter de répondre en étudiant les modèles cosmologiques de Bianchi ou autrement formulé: partant d'un modèle cosmologique homogène anisotrope dont les modèles FLRW sont un sous-ensemble, quelles propriétés doivent avoir les théories tenseurs-scalaires afin que ces modèles possèdent asymptotiquement les caractéristiques dynamiques de notre Univers actuel ou apportent une réponse à certains de ses problèmes comme ceux de la constante cosmologique?

Evidement la question est vaste et il serait illusoire de penser pouvoir y répondre complètement ou définitivement. Elle doit avant tout nous servir de fil conducteur nous menant vers quelques éléments de réponses. Le chemin que nous avons suivi s'est déroulé en deux étapes.
La première a consisté à se demander quelles caractéristiques dynamiques (expansion asymptotique, absence de singularité, présence de symétries, etc...) souhaiterions nous que les modèles homogènes possèdent

et à explorer un certain nombres de méthodes permettant de connaître et de contraindre efficacement les théories tenseur-scalaires afin qu'elles soient compatibles avec ces caractéristiques. La plupart des méthodes de cette première étape étaient limitées à certain types de modèles cosmologiques ou de théories tenseur-scalaires et ne permettaient pas une approche globale et unifiée de l'ensemble de ces deux composantes. Finalement, nous avons trouvé qu'il était possible de contraindre un grand nombre de modèles homogènes et de théories tenseur-scalaires en conjuguant le formalisme Hamiltonien ADM avec les méthodes d'analyse des systèmes dynamiques afin d'étudier le processus d'isotropisation de ces modèles. C'est là le commencement de la deuxième étape de notre travail: l'application systématique de cette méthode à tous les modèles de Bianchi de la classe A et en considérant des théories tenseur-scalaires possédant jusqu'à trois fonctions inconnues du champ scalaire [1]. Ceci nous a permis de classifier les théories tenseur-scalaires en trois classes en fonction de leur mode d'isotropisation. Notre attention s'est entièrement portée sur l'une d'entre elle étroitement liée aux théories de quintessence et nous avons pû alors déterminer les comportements asymptotiques des modèles de Bianchi au voisinage de l'isotropie.

Dans les deux chapitres suivants, on expose respectivement l'intérêt des champs scalaires et des modèles de Bianchi à travers une argumentation historique et physique.

1. Comme nous le verrons plus tard, une fonction de gravitation, une fonction de Brans-Dicke et un potentiel

Chapitre 2

L'intérêt des champs scalaires

2.1 La naissance de la première théorie tenseur-scalaire

Les champs scalaires ont une longue histoire comme le montre l'article de H. Brans[1] dont nous nous sommes servis pour écrire cette section. Celle-ci commence par une tentative d'intégration de la théorie newtonienne et de son potentiel scalaire dans une théorie de la relativité restreinte que nous n'aborderons pas ici, en passant par les théories de Kaluza-Klein et les nombres de Dirac pour aboutir à la première théorie tenseur-scalaire de Jordan, Brans et Dicke. C'est la naissance de cette dernière que nous allons raconter.
L'objectif des théories de Kaluza-Klein était d'unifier la gravitation avec l'électromagnétisme. Pour cela l'idée est d'introduire le 4-potentiel électromagnétique dans la métrique en rajoutant une cinquième dimension à l'espace-temps[2]: la courbure en un point de l'espace-temps désormais à 5 dimensions y engendre ce que l'on perçoit comme étant les forces gravitationnelles et électromagnétiques. Cette cinquième dimension est compactifiée à l'échelle de Planck et est donc inobservable.
Où se cache le champ scalaire de cette théorie? Si l'on considère des indices (M,N) variant de 0 à 4 et des indices (μ,ν) variant de 0 à 3, la 5 métrique de Kaluza-Klein définie par les fonctions g_{MN} est composée de:

- la 4-métrique habituelle, représentée par les fonctions métriques $g_{\mu\nu}$
- le 4 potentiel électromagnétique qui est contenu dans les fonctions métriques $g_{\mu 4} = g_{4\mu}$
- et enfin une composante g_{44} choisie constante.

La constance de la fonction g_{44} est une hypothèse de Kaluza qui fut plus tard abandonnée apparemment en premier[3] par Jordan[4] puis par Thiry[5]: il montra que la composante g_{44} correspondait en fait à un champ scalaire. Pour cela, il écrivit l'ensemble complet des équations de champs pour le tenseur de Ricci, $R_{MN} = 0$ qui se réduit alors à 10 équations d'Einstein avec matière, 4 équations de Maxwell et une équation d'onde pour le champ scalaire, cette dernière n'ayant rien à voir avec la gravitation ou l'électromagnétisme. Pour retrouver la théorie d'Einstein-Maxwell standard, on s'aperçoit qu'il faut alors choisir de manière ad hoc $g_{44} = 4G$, où G est la constante de gravitation, associant ainsi le champ scalaire à cette constante.
C'est alors qu'intervint Dirac[6]. A partir de l'âge de l'Univers T_u tel que défini par les mesures de la constante de Hubble en 1938 et d'une échelle de temps atomique T_a naturellement définie par les échelles de temps e^2/m ou $\hbar/m$, où e est la charge de l'électron et m est la masse d'un électron ou d'un nucléon, il définit le rapport de temps $t \equiv T_u/T_a \approx 10^{40}$. Puis, Dirac décide de regarder le rapport sans dimension des forces électriques et gravitationnelles. Il définit alors le nombre $\gamma \equiv e^2/(km^2) \approx 10^{40}$ avec $k = 8\pi G$. Il définit également le rapport entre la masse de l'Univers M_u et une masse atomique standard m soit $\mu \equiv M_u/m \approx 10^{80}$. Pour Dirac, la manière dont ces nombres naturels et sans dimension se regroupent doit avoir une raison physique qui le conduit à développer un modèle cosmologique pour lequel $\mu \approx t^2$ et $\gamma \approx t$, c'est-à-dire tel que ces deux quantités varient avec le temps, impliquant ainsi que $\mu/(t\gamma) \approx 1$ et donc

$$1/k \approx M/R \tag{2.1}$$

M et R étant la masse et le rayon de l'Univers. Cette dernière égalité soulève alors la question de savoir si la constante de gravitation est une vraie constante ou si elle est déterminée par la distribution de masse dans l'Univers.
L'association du champ scalaire des théories de Kaluza-Klein à la constante gravitationnelle et la possible variation de cette dernière due à l'hypothèse (2.1) de Dirac, firent penser à Jordan que le champ scalaire pourrait être une généralisation d'une constante de gravitation qui serait en fait variable. Brans et Dicke motivés par les idées de Mach sur l'inertie ainsi que Sciama, arrivèrent à des conclusions similaires sur

une possible variation de G. Cependant ce fut Jordan et ses collaborateurs qui firent les premiers un pas supplémentaire décisif en séparant le champ scalaire de son contexte multi dimensionnel. Dans toutes ces théories, le champ scalaire ϕ vaut approximativement l'inverse de la constante de gravitation:

$$1/k \approx \phi$$

Ce choix est motivé par l'hypothèse (2.1) qui montre que $1/k$ pourrait être une variable et satisfaire une équation de champ. Si maintenant on écrit l'action de la Relativité Générale, il vient

$$\delta \int (R + kL_m)\sqrt{-g}d^4x = 0$$

Le couplage de k, quantité variable, directement au Lagrangien de la matière L_m, fait que les particules ne suivent plus les géodésiques de l'espace-temps en l'absence de toute autre force que la force gravitationnelle. Afin de remédier à ce problème, on divise l'action par k et on obtient finalement:

$$\delta \int (\phi R + L_m)\sqrt{-g}d^4x = 0$$

L'équation des géodésiques est donc sauve mais le champ scalaire modifie évidemment l'énergie du champ de gravitation et implique des effets observables. L'action ci-dessus n'est pas encore satisfaisante. En effet, elle ne donne pas lieu à une équation pour ϕ qui nous permettrait de connaître sa dynamique. Pour cela, il nous faudrait une action de la forme:

$$\delta \int (\phi R + L_\phi + L_m)\sqrt{-g}d^4x = 0$$

et puisque les équations de champs sont habituellement du second ordre, il est probable que $L_\phi = L(\phi,\phi_{,\mu})$. Un choix naturel semble être $L_\phi = -\omega\phi_{,\mu}\phi_{,\nu}g^{\mu\nu}$, où ω est une constante. Cependant ω devrait avoir la même dimension que la constante de gravitation et le choix final est donc

$$L_\phi = -\frac{\omega}{\phi}\phi_{,\mu}\phi_{,\nu}g^{\mu\nu}$$

Nous obtenons ainsi la forme de l'action de la théorie de Jordan-Brans-Dicke[7], la première théorie tenseur-scalaire:

$$\delta \int (\phi R - \frac{\omega}{\phi}\phi_{,\mu}\phi_{,\nu}g^{\mu\nu} + L_m)\sqrt{-g}d^4x = 0$$

On peut alors appliquer le principe variationnel sur cette action afin de trouver les équations de champs dont l'équation pour le champ scalaire. Ainsi, pour un champ faible et pour une coquille sphérique de masse M et de rayon R, l'Univers étant vide de par ailleurs, cette équation donne:

$$\phi \approx \phi_\infty + \frac{1}{4\pi(2\omega + 3)}\frac{M}{R}$$

Si ϕ est identifié avec l'inverse de la constante de gravitation et que ϕ_∞ est choisi égal à zéro, on retrouve l'hypothèse de Dirac (2.1).

Le début des années 80 a profondément modifié les raisons de considérer des champs scalaires: les idées de Guth sur l'inflation donnèrent naissance à des champs scalaires appelés inflatons tandis que l'émergence de nouvelles idées en physique des particules donnèrent naissance aux dilatons qui seront abordés dans la section suivante. Le modèle d'alors de la cosmologie souffre de nombreux problèmes conceptuels: pourquoi l'Univers semble t'il si plat? Comment des régions causalement séparées au début des temps peuvent elles être si semblables aujourd'hui? Guth[8] remarqua qu'ils seraient partiellement résolus si, aux époques primordiales, il y avait une période d'inflation avec une expansion exponentielle de l'Univers. Pour cela, la première idée est d'introduire une constante cosmologique mais les observations montrent que sa valeur actuelle serait 10^{120} fois plus petite que celle prédite aux époques primordiales: c'est le problème de la constante cosmologique. Une manière de le résoudre est de considérer un nouveau champ scalaire appelé inflaton tel que

$$L_\phi = \phi_{,\mu}\phi_{,\nu}g^{\mu\nu} - U(\phi)$$

dont le couplage avec lui même est décrit par le potentiel U qui joue alors le rôle d'une constante cosmologique variable. Depuis la fin des années 90, la présence de ce potentiel a trouvé de nouvelles raisons d'être avec la détection par deux équipes[9, 10] indépendantes de l'accélération de l'expansion de l'Univers. L'une des explications les plus en vogue de ce phénomène serait la présence d'un champ scalaire quintessent, c'est-à-dire dont la densité et la pression sont liées par une équation d'état semblable à celle d'un fluide parfait et dont l'indice barotropique serait négatif. Il en résulterait une pression du champ scalaire négative qui serait à l'origine de cette nouvelle et récente période d'accélération de l'expansion.

2.2 Les champs scalaires en physique des particules

L'introduction de champs scalaires en cosmologie obéit également à des raisons profondes liées à la physique des particules. Afin de les appréhender, nous allons en exposer quelques points importants. Cette section s'inspire d'un article de Zel'dovich[11] destiné à vulgariser le concept de champs scalaire.
Les théories physiques les mieux établies par l'expérimentation reposent sur des champs vectoriels et tensoriels. Un champ de vecteurs est une distribution spatio-temporelle de 4-vecteurs: à chaque point de l'espace en chaque instant est associé un vecteur. Citons quelques champs de vecteurs couramment utilisés en physique et aux propriétés très différentes:

- Le plus évident est bien sûr le champ électromagnétique. Ce champ de vecteurs est neutre (le photon n'a pas de charge) et non massif et cette interaction est donc à porté infinie.
- Le champ de vecteurs des bosons W et Z responsables de l'interaction faible est massif, ce qui signifie qu'elle est à courte portée et instable: ces particules se désintègrent en paires d'autres particules.
- Le champ de vecteurs des gluons, responsable de l'interaction forte, est massif et avec une charge. Les gluons sont eux même une source pour d'autres champs de gluons menant au confinement des quarks et au fait que les gluons comme les quarks ne peuvent exister librement. Les seules particules stables sont ainsi des combinaisons de quarks et d'antiquarks ou des combinaisons de trois quarks.

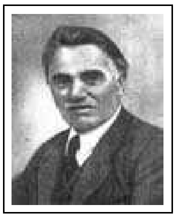
Il existe d'autres types de champs que les champs vectoriels. Ainsi dans la liste ci-dessus ne figure pas la description de la force gravitationnelle qui ne repose pas sur un champ de vecteurs mais sur un champ de tenseurs. Comme on le voit, tous ces champs correspondent à des particules: ainsi les champs de vecteurs correspondent à des particules avec un spin $1\hbar$, $\hbar$ étant la constante de Plank divisée par 2π, et les champs de tenseurs à des particules (gravitons) de spin $2\hbar$. Les particules de spin entier sont des bosons et, contrairement à celles ayant un spin demi entier et qui sont des fermions, elles n'obéissent pas au principe d'exclusion de Pauli. Les champs scalaires quant à eux correspondent à des particules de spin zéro et sont donc également des bosons.
Historiquement, la première raison d'introduire un champ scalaire en physique des particules est due à Yukawa, qui imagina un champ avec une masse au repos afin d'expliquer les forces nucléaires. La courte portée des interactions nucléaires correspondait à des mésons ayant une masse de l'ordre de 100 à 200 Mev prédite par Yukawa. On découvrit effectivement des mésons π avec une masse de 140 Mev. L'avenir des champs scalaires semblait donc tout tracé. Cependant, les prédictions détaillées de la théorie scalaire furent désavouées par l'expérience. Les pions étaient bien des bosons mais composés d'un quark et d'un antiquark. Le renouvellement de l'intérêt pour les champs scalaires vint d'une idée complètement différente de celle de Yukawa: la renormalisation.
Le développement de l'électrodynamique quantique commença vers la fin des années 40. L'approximation de base pour les muons et les électrons dans un atome ou un champ magnétique est donnée par la théorie de Dirac. Les électrons ont un spin, une charge et un moment définis. Cependant, pour une meilleure approximation, il est nécessaire de tenir compte de processus virtuels: un électron dans son état de base ne peut émettre un photon réel car il n'a pas l'énergie requise. Mais il peut émettre un photon virtuel puis le réabsorber rapidement. Ceci est permis par le principe d'incertitude d'Heisenberg à partir du moment où la conservation de l'énergie n'est pas violée. De la même manière, on peut imaginer la création et l'annihilation de paires virtuelles d'électron-positron dans le champ électrostatique du noyau. Ces processus ne changent pas qualitativement la physique: il y a toujours un état de base de l'électron et un champ électrostatique. Mais les propriétés quantitatives sont très légèrement changées lorsque l'on prend en compte ces processus virtuels: à première vue, les équations semblent contenir une infinité d'intégrales à cause du nombre infini d'états possibles des particules virtuelles. Cependant, il fut réalisé que des processus similaires arrivaient pour les électrons libres comme pour les liés, que la quantité mesurée est la différence entre les énergies de ces deux types d'électrons et que la différence entre deux intégrales infinies est une intégrale finie. Cette procédure fut appelée renormalisation et montra l'importance des particules virtuelles.
Par la suite, on commença également à s'intéresser aux corrections du second ordre pour la force faible mais cette fois, la renormalisation ne marcha pas pour les particules massives vecteurs de cette force: les infinis ne disparaissaient pas. Notons que cette masse est nécessaire pour expliquer la désintégration β et le rôle des particules W et Z. C'est là que l'idée des champs scalaires revint en force. Plutôt que d'imposer directement une masse aux bosons, on suppose que le champ de vecteurs les représentant interagit avec la charge d'un champ scalaire massif, c'est-à-dire possédant un potentiel, la charge décrivant l'interaction avec le champ de vecteurs. Ce champ scalaire est le champ de Higgs-Englert qui eurent l'idée d'introduire un potentiel de la forme $V(\phi) = k(\phi^2 - \phi_0^2)$, permettant ainsi la renormalisation. Ce champ seulement caractérisé par une masse et qui correspond comme expliqué plus haut à un boson, donne leur masse aux particules qui interagissent avec lui. On espère une détection de la particule de Higgs dans le futur LHC vers 2008.

D'autres raisons peuvent être évoquées concernant la présence de champs scalaires de nature différente de celle du boson de Higgs. Sans rentrer autant que précédemment dans les détails signalons par exemple que les théories de supersymétries prédisent l'existence de plusieurs champs scalaires. Ces théories postulent l'égalité entre les degrés de liberté fermionique et bosonique: à chaque boson (dont celui de Higgs) correspond un fermion et vice-versa. Ceci ne peut être réalisé qu'en ajoutant des degrés de libertés supplémentaires via des champs scalaires dont les potentiels peuvent être tout à fait différents de celui imaginé par Higgs (par exemple des champs scalaires complexes). Généralement ces champs scalaires sont appelés dilatons. Les particules supersymétriques pourrait être des constituants essentiels de la matière noire. En particulier le plus léger des neutralinos, un état résultant d'une mixture de plusieurs particules supersymétriques, pourrait être la plus légère des particules antisymétriques et un candidat pour la théorie de la matière noire froide. Pour une introduction aux théories de supersymétrie, on pourra se référer au livre de Gordon Kane[12], "Supersymmetry, unveiling the ultimate laws of Nature".
Tout ceci démontre, nous l'espérons, l'intérêt de considérer des champs scalaires.

Chapitre 3

L'intérêt des modèles de Bianchi

3.1 Un peu d'histoire



Luigi Bianchi[13] est né à Parme le 18 juin 1856. Il fut l'étudiant d'Ulisse Dini et Enrico Batti à l'école normale supérieure de Pise et devint professeur à l'Université de Pise en 1886 puis directeur de l'école normale en 1918 jusqu'à sa mort en en 1928.
Ses contributions mathématiques furent publiées dans 11 volumes par l'Italian Mathematical Union et couvrent un grand nombre de domaines. En ce qui concerne la géométrie Riemanienne, il est surtout connu pour sa découverte des identités de Bianchi dont la démonstration complète fut donnée dans [14](il les avait découvertes une première fois dans un article de 1888 mais avait négligé leur importance en les donnant en note de bas de page.). En 1897, en utilisant les résultats de Lipshitz[15] et Killing[16] ainsi que la théorie des groupes continus de Lie[17], il donna une classification complète des classes d'isométries des 3-variétés de Riemann, identifiées par les lettres romaines I à IX. A l'époque, ni la relativité générale, ni la relativité restreinte n'existaient encore [1].
En 1951, le travail de Bianchi fut introduit en cosmologie par Abraham Taub dans son article "Empty Spacetimes Admitting a Three-Parameter Group of Motions"[18]. Les espaces temps spatialement homogènes ont une géométrie spatiale dépendante du temps qui est donc une 3-géométrie homogène. Ainsi, l'espace temps a un groupe d'isométries à r dimensions agissant sur une famille d'hypersurfaces avec $r = 3$(action simplement transitive), $r = 4$(locally rotationally symmetric) ou $r = 6$(modèles isotropiques). Le cas $r = 3$ devint connu sous le nom de cosmologies de Bianchi après l'article de Taub.
Pendant une décade, les modèles de Bianchi tombèrent dans l'oubli jusqu'à la renaissance de la Relativité Générale au début des années 60. O. Heckmann et E. Schücking les firent resurgir en 1958 dans leur ouvrage "Gravitation, an Introduction to Current Research"[19]. Puis ce fut au tour de l'école russe de Lifshitz et Khalatnikov, rejoints plus tard par Belinsky, à travers leur étude sur l'approche chaotique de la singularité initiale qui inspira Misner aux USA et plus tard Hawking et Ellis au Royaume Uni. La classification de Bianchi elle-même fut revue par C.G. Behr dans un travail non publié mais rapporté dans [20] en 1968. Enfin une contribution essentielle sur les modèles de Bianchi fut apportée par Ryan, un étudiant de Misner, et résumée à travers son livre "Homogenous Relativistic Cosmologies"[21].

3.2 Un peu de physique

L'Univers tel que nous l'observons aujourd'hui est très bien décrit par les modèles cosmologiques et homogènes de Friedman-Lemaitre-Robertson-Walker (FLRW). Ceci a été montré par les observations du rayonnement de fond cosmologique par les satellites COBE et WMAP. Cependant rien ne nous permet d'extrapoler ces propriétés d'isotropie et d'homogénéité aux époques primordiales avant le découplage rayonnement/matière. La question se pose donc de savoir pourquoi l'Univers les possède alors qu'il existe une infinité de modèles cosmologiques ne les ayant pas. Essentiellement, on distingue deux réponses:

- Il pourrait exister un principe quantique qui sélectionne parmi l'ensemble des modèles possibles, les modèles FLRW comme étant les plus probables. Cette réponse repose sur le développement d'une théorie quantique des conditions initiales.
- L'univers primordial serait inhomogène et anisotrope mais son évolution le conduirait vers un état (asymptotique ou temporaire) homogène et isotrope correspondant aux modèles FLRW.

1. Il existe également le modèle anisotrope axisymétrique de Kantowski et Sachs

C'est ce second point de vue que nous adopterons en considérant que l'Univers est initialement anisotrope et devient asymptotiquement isotrope. Nous garderons l'hypothèse d'homogénéité car les modèles cosmologiques inhomogènes ne sont pas classifiés, dus à leur manque de symétrie. De plus nous considérerons que cet état n'est pas transitoire mais atteint asymptotiquement. En effet, les observations montrent que notre Univers doit être isotrope depuis au moins son premier million d'années ce qui laisse à penser que cet état une fois atteint, est stable.
Considérer que l'Univers est initialement anisotrope permet donc d'expliquer les processus menant à l'isotropisation plutôt que de considérer cet état de manière ad hoc. Un autre avantage des modèles anisotropes réside en leur approche de la singularité en Relativité Générale, oscillatoire et chaotique pour les plus généraux d'entre eux. Elle serait partagée, selon une conjoncture due à BKL, par les modèles inhomogènes et anisotropes au contraire des modèles FRLW dont l'approche de la singularité est monotone.

Chapitre 4

Plan du travail

Le reste de cette thèse est divisé en quatre parties.

Dans la partie II, nous nous intéresserons aux outils mathématiques qui nous permettront d'établir les équations de champs. Pour cela, nous commencerons par classifier les modèles de Bianchi afin d'établir leurs métriques. Puis nous étudierons la méthode de Cartan qui permet de déterminer rapidement les composantes non nulles du tenseur de Riemann et donc d'obtenir le tenseur d'Einstein. Etant ainsi capable de déterminer la partie géométrique des équations de champs nous considérerons un contenu matériel pour l'Univers en écrivant le Lagrangien des théories tenseur-scalaires dont nous déduirons la forme complète des équations de champs pour tous les modèles de Bianchi. Enfin, nous ferons de même à l'aide du formalisme Hamiltonien ADM.

Dans la partie III, nous nous servirons de plusieurs méthodes permettant d'étudier la dynamique des modèles homogènes en théories tenseur-scalaires afin de comprendre ce qu'elles peuvent nous apprendre mais aussi quelles sont leurs limites. Nous commencerons par ce qui est conceptuellement (mais pas forcément techniquement) le plus simple, c'est-à-dire la recherche de solutions exactes. Puis nous verrons comment l'on peut analyser le comportement asymptotique des fonctions métriques. Enfin nous montrerons deux études basées sur l'exigence de l'absence de singularité ou la présence d'une symétrie de Noether. Cette partie est entièrement composée d'articles publiés que nous avons ici reproduits dans leur intégralité.

La partie IV, est consacrée à l'isotropisation des modèles de Bianchi en présence d'un ou de plusieurs champs scalaires. De toutes les méthodes testées, c'est sans aucun doute celle qui nous permet d'étudier le plus vaste ensemble de modèles et de théories à l'aide d'un formalisme unifié et basé sur une exigence physique solide, soit la nécessité pour l'Univers de s'isotropiser. Cette partie synthétise un ensemble d'articles dont ceux déjà publiés sont reproduits en annexe. L'aspect énergie noire et matière noire des champs scalaires est alors abordé à travers respectivement la quintessence et l'aplatissement des courbes de rotations des galaxies spirales.

Enfin dans la partie V, on discute et on conclut.

Deuxième partie

Outils mathématiques

Chapitre 1

Les modèles de Bianchi

Dans ce chapitre nous allons étudier la classification des modèles spatialement homogènes et isotropes de Bianchi. Nous verrons qu'ils sont au nombre de neuf et se subdivisent en deux classes, A et B. Enfin nous apprendrons à calculer la métrique de chacun de ces modèles.

1.1 Classification des variétés spatialement homogènes de Bianchi

Les modèles de Bianchi sont des variétés spatialement homogènes mais anisotropes que l'on peut classifier à l'aide des groupes et algèbres de Lie[21, 22, 23] comme nous allons le voir dans cette section.

1.1.1 Quelques définitions

Pour commencer, définissons ce qu'est un ***groupe topologique***. Un groupe topologique est un groupe G munit d'une topologie qui rend continues les applications suivantes:

– La loi de composition de G: $(a,b) \to ab$
– L'inversion de G: $aob^{-1} = e$ où e est l'élément inverse.

Un espace est dit connexe si 2 points quelconques de cet espace peuvent être reliés par une courbe déformable à volonté d'une façon continue, telle que tous les points de la courbe se trouvent à l'intérieur de l'espace.
Un groupe de Lie est alors la composante connexe d'un groupe topologique.

Définissons le ***commutateur*** $[X,Y]$ de deux champs de vecteurs quelconques X et Y. Soit une fonction quelconque ψ, on a:

$$[X,Y]\,\psi = X(Y\psi) - Y(X\psi)$$

Une ***isométrie d'une variété riemannienne*** M est une transformation de M qui laisse la métrique g invariante. Les isométries de la variété M forment un groupe de transformations de M. Elles conservent les mesures des longueurs, les mesures d'angles et transforment les géodésiques en géodésiques. L'ensemble de toutes les isométries d'une variété M donnée vérifie les axiomes de groupe car l'identité est une isométrie, l'inverse d'une isométrie est une isométrie et la composition de deux isométries est encore une isométrie. Cette ensemble forme donc lui même un groupe, généralement ***un groupe de Lie***.
Cette définition du commutateur va nous permettre de définir ce qu'est une ***algèbre de Lie***. Une algèbre de Lie réelle L, de dimension $n \geq 1$, est un espace vectoriel réel de dimension n muni d'un produit de Lie [,] tel que $\forall a,b,c \in L$ et $\forall \alpha,\beta,\gamma \in \Re$:

– $[a,b] \in L$
– $[\alpha a + \beta b,c] = \alpha\,[a,c] + \beta\,[b,c]$
– $[a,\beta b + \gamma c] = \beta\,[a,b] + \gamma\,[a,c]$
– $[a,b] = -\,[b,a]$ (antisymétrie)
– $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$ (identité de Jacobie)

Une algèbre de Lie est spécifiée par une base $x_1,...,x_n$ de l'espace vectoriel de cette algèbre. Puisque le produit de Lie de deux éléments de base appartient encore à l'algèbre de Lie, on peut écrire

$$[x_i,x_j] = C^k_{ij}x_k$$

On défini ainsi les ***coefficients de structure*** de l'algèbre de Lie.
Les isométries sont générées par ce que l'on appelle les ***vecteurs de Killing*** ξ. Ils sont tels qu'ils vérifient les ***équations de Killing***

$$\xi_{a;b} + \xi_{b;a} = 0$$

Tout vecteur de Killing engendre une isométrie et l'ensemble de tous ces vecteurs forme l'algèbre de Lie du groupe d'isométrie. Par conséquent, rechercher les isométries d'une variété consiste à rechercher les solutions des équations de Killing. Par exemple, considérons un espace minkowskien dont l'élément de longueur quadridimensionnel infinitésimal s'écrit $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$. Les équations de Killing fournissent dix vecteurs de Killing indépendants. Ce sont en fait les générateurs infinitésimaux du groupe de Poincaré correspondant aux 4 générateurs des translations spatio-temporelles, trois générateur des rotations à trois dimensions et trois générateurs des transformations homogènes de Lorentz. On a donc un groupe G_{10} agissant sur une variété M_4 représentant une variété possédant un nombre maximum de symétries. De ce fait, l'espace temps de Minkowski est à courbure de Riemann constante. De manière générale, un groupe d'isométrie G_r à r paramètres agissant sur une variété M_n à n dimensions est telle que $n \leq r \leq n(n+1)/2$. La variété M_n est alors dite à ***symétrie maximale*** lorsque $r = n(n+1)/2$.
Pour imposer l'homogénéité spatiale, nous avons donc besoin d'un groupe d'isométrie agissant transitivement sur les sections spatiales ($n = 3$) de l'espace temps. ***Un groupe est transitif*** sur une surface S quelque soit sa dimension si il peut transformer n'importe quel point de S en un autre point de S. Il existe donc quatre groupes d'isométrie possible car $3 \leq r \leq 6$, soient G_6, G_5, G_4 et G_3. Le groupe G_6 de symétrie maximale correspond aux modèles homogènes et isotropes de la classe des FLRW. Le groupe G_5 est interdit par le ***théorème de Fubini*** qui affirme qu'une variété riemannienne de dimension supérieure à deux et qui n'est pas à courbure riemannienne constante, admet au plus un groupe d'isométrie à $n(n+1)/2 - 1$ paramètres. Le groupe G_4 peut toujours se ramener, sauf dans le cas du modèle de Kantowski-Sachs, au groupe G_3 car il admet toujours, sauf dans un cas, un sous groupe à trois paramètres agissant simplement transitivement sur des hypersurfaces spatiales.
Par conséquent, ***à part le modèle de Kantowski-Sachs, la classification de tous les modèles d'Univers homogènes se ramène à celle des groupes d'isométrie spatiale à trois paramètres, soit les algèbres de Lie réelles à trois dimensions.***

1.1.2 La classification des algèbres de Lie réelles à trois dimensions

Soit une base ξ_λ, $\lambda = 1,2,3$ de l'algèbre de Lie telle que $[\xi_\lambda,\xi_\mu] = C^\nu_{\lambda\mu}\xi_\nu$. Les commutateurs étant antisymétriques et vérifiant les identités de Jacobi, on a $C^\nu_{(\lambda\mu)} = 0$ et $C^\nu_{[\lambda\mu}C^\sigma_{\rho]\nu} = 0$ ce qui réduit à 9 le nombre de constantes de structure indépendantes. On peut réécrire celles ci à l'aide de la ***décomposition d'Elis et MacCallum*** faisant intervenir un pseudotenseur symétrique $n^{\lambda\mu}$ et un vecteur a_μ:

$$C^\nu_{\lambda\mu} = \epsilon_{\sigma\lambda\mu} n^{\nu\sigma} + 2\delta^\nu_{[\mu} a_{\lambda]}$$

où les δ sont les symboles de Kroenecker et les ϵ les symboles de Levi-Civita tels que, en coordonnées de Minkowski, $\epsilon_{\sigma\lambda\mu} = -\epsilon^{\sigma\lambda\mu}$ et $\epsilon_{123} = 1$. Les crochets indiquent l'opération d'antisymétrisation sur les indices qu'ils renferment. On en déduit que

$$a_\mu = \frac{1}{2} C^\nu_{\mu\nu}$$

$$n^{\lambda\mu} = \frac{1}{2} C^{(\lambda}_{\sigma\tau} \epsilon^{\mu)\sigma\tau}$$

Les parenthèses indiquent l'opération de symétrisation sur les indices qu'ils renferment. Cette décomposition vérifie la propriété d'antisymétrie et les identités de Jacobi fournissent

$$n^{\lambda\mu} a_\mu = 0$$

C'est cette équation aux valeurs et vecteurs propres qu'il faut résoudre pour trouver toutes les structures possibles d'une algèbre de Lie de dimension trois et donc les solutions qui ne sont pas mutuellement équivalentes par un quelconque changement de base ξ_λ. La matrice $n^{\lambda\mu}$ est symétrique et réelle et on peut donc appliquer le théorèmes de JJ.Sylvester qui nous dit que le rang 1 et la valeur absolue $| s |$ de sa signature [1] sont invariants sous l'action d'un changement de base. Il faut donc chercher les diverses combinaisons possibles de rang et de signature de la matrice $n^{\lambda\mu}$. On scinde les modèles de Bianchi en 2 classes.

– ***La classe*** A ***de Bianchi*** est telle que $a_\lambda = 0$.
  On choisi une base dans laquelle le tenseur $n^{\lambda\mu}$ est diagonal et dont les valeurs propres $n^{(i)}$, éléments

1. La valeur absolue de la somme des ´el´ements diagonaux

Classe	Type	$n^{(1)}$	$n^{(2)}$	$n^{(3)}$	a	dimension
A	I	0	0	0	0	0
A	II	1	0	0	0	3
A	VI_0	1	-1	0	0	5
A	VII_0	1	1	0	0	5
A	$VIII$	1	1	-1	0	6
A	IX	1	1	1	0	6
B	V	0	0	0	1	3
B	IV	1	0	0	1	5
B	$III = VI_{-1}$	1	-1	0	1	5
B	$VI_h(h < 0)$	1	-1	0	$\sqrt{-h}$	(6)5 si h fixé
B	$VII_h(h > 0)$	1	1	0	$\sqrt{h}$	(6)5 si h fixé

TAB. 1.1 – *Classification des algèbres de Lie*

diagonaux de $n^{\lambda\mu}$, valent 0,1 ou -1. On a alors six manières de combiner le rang et la signature de la matrice $n^{\lambda\mu}$ correspondant à six modèles: I, II, VI_0, VII_0, $VIII$ et IX.

– ***La classe** B **de Bianchi*** est telle que $a_\lambda \neq 0$.
Dans ce cas a_λ est vecteur propre de $n^{\lambda\mu}$ relativement à la valeur propre 0. On choisi une base dans laquelle le tenseur $n^{\lambda\mu}$ est diagonal avec les valeurs propres $n^{(i)}$ et telle que les vecteurs a_λ soient orientés le long du troisième axe. On en déduit que $n^{(3)} = 0$ car $a \neq 0$ et donc que le rang de la matrice est inférieur ou égal à deux. Il existe donc quatre combinaisons possibles de rang et de signature de la matrice $n^{\lambda\mu}$. Si de plus on utilise la transformation d'échelle $\xi_i = k_i \xi'_i$ avec k_i une constante, on montre que la quantité $h^{-1} = n^{(1)} n^{(2)} a^{-2}$ est un invariant. Les quatre modèles seront nommés: IV, V, VI_h et VII_h.

– ***Dimension des algèbres de Lie***
Chaque classe d'équivalence des constantes de structure $C^\nu_{\lambda\mu}$ de l'algèbre de Lie constitue une sous-variété de l'espace des tenseurs à trois indices et donc de dimensions 27. Les constantes de structure étant antisymétriques (27-18=9) et respectant les trois identités de Jacobi (9-3=6), chaque type de Bianchi est donc associé à une sous variété de dimension six au maximum. Pour les types de la classe A, ceci correspond aux six composantes de la matrice symétrique $n^{\lambda\mu}$, pour les types de la classe B, aux trois composantes du vecteur a_μ et aux trois composantes de $n^{\lambda\mu}$ dans le plan perpendiculaire au troisième axe. On en déduit que:
  – Pour les types VI_h, VII_h, $VIII$ et IX de Bianchi, il n'y a aucune restriction et il existe des ensembles de constantes de structure de dimension maximale égale à six.
  – Pour les types VI_h et VII_h, si h est fixé, on a une contrainte et donc leurs ensembles de constantes de structure sont de dimension cinq.
  – Pour les types II et V, un vecteur est donné (a_μ pour V et la première ligne de $n^{\lambda\mu}$ pour II). Leurs ensembles de constantes de structure sont donc de dimension trois.
  – Pour le type I, les constantes de structure sont toutes nulles et donc la dimension de leurs ensembles est zéro.

L'ensemble de cette classification est résumé dans le tableau 1.1.

1.2 Les métriques des variétés spatialement homogènes de Bianchi

Une congruence est un ensemble de courbes remplissant complètement au moins une région localement délimitée de la variété considérée. Pour écrire une métrique, il nous faut choisir une congruence temporelle et une base spatiale.

1.2.1 Congruence temporelle

Soit un ensemble d'hypersurfaces spatiales invariantes sous l'action des éléments d'un groupe d'isométries $G_{r\geq 3}$. Soit S, l'une des surfaces et un point P appartenant à S. On trace la géodésique temporelle normale à S et passant par P. n^α est le vecteur unitaire tangent à cette géodésique le long de laquelle on mesure une distance propre s. On obtient alors un point Q et on construit ainsi la surface S' à laquelle ce point appartient. Soit P', un autre point quelconque de S, comme le groupe d'isométries est transitif, il existe une transformation $\phi \in G_r$ telle que $\phi(P) = P'$. A nouveau $Q' \in S'$ se déduit de P' en portant la même

distance s le long de la géodésique temporelle perpendiculaire à S et passant par P'. On engendre ainsi un espace tangent aux hypersurfaces spatiales invariantes par G_r.
Soient $\xi_{(m)}$, $m = 1...r$, les vecteurs de Killing qui engendrent en tous les points de l'espace temps, l'espace tangent aux hypersurfaces spatiales invariantes. Les vecteurs $\xi_{(m)}$ obéissent aux équations de Killing $\xi_{(m)\alpha;\beta} + \xi_{(m)\beta;\alpha} = 0$ et n^α obéit à l'équation des géodésiques $n^\alpha_{;\beta}n^\beta = 0$. On en déduit que $n^\alpha\xi_{(m)\alpha} = 0$ et donc que *la géodésique temporelle de vecteur tangent n^α est orthogonale à toute surface homogène qu'elle coupe* car $n^\alpha \perp \xi_{(m)\alpha} \forall m = 1...r$. Par conséquent *les normales aux hypersurfaces d'homogénéité constituent le champ de vecteurs tangents d'une congruence de géodésique du genre temps, orthogonales à des hypersurfaces spatiales*. On choisit alors la direction de n^α pour définir la variable temporelle t. Les hypersurfaces spatiales homogènes sont alors des surfaces S(t) où t reste constant. Ces surfaces sont paramétrées par la distance mesurée le long des géodésiques temporelles, d'où $n_\alpha = -\partial t/\partial x^\alpha = (-1,0,0,0)$. Ce choix fixe un référentiel synchrone avec $g_{00} = -1$ et $g_{0m} = 0 \forall m = 1,2,3$. $x^0 = t$ est le temps propre de chaque point de l'espace et la métrique d'un espace temps dans le référentiel synchrone s'écrit $ds^2 = -dt^2 + g_{mn}dx^m dx^n$, $m,n = 1,2,3$. Comme on le montrera au paragraphe suivant, il n'y a pas de mélange des variables spatiales et temporelles. Du fait de l'homogénéité spatiale, le champ de vecteurs n^α est invariant sous l'action des éléments du groupe G_r. Cette invariance de groupe implique l'annulation de sa dérivée de Lie relativement à n'importe quel générateur infinitésimal des isométries. Il s'ensuit que n^α commute avec tous les vecteurs de Killing:

$$\left[\xi_{(\mu)}, n\right] = 0$$

1.2.2 Base spatiale

Soit un groupe de transformations infinitésimales G_r et une base de vecteurs de Killing $(\xi_{(\mu)})$. On définit l'orbite d'un groupe en un point P de la variété M comme étant une sous variété de M constituée des points de M qui résultent de l'action de tous les éléments du groupe sur le point P. On va rechercher l'ensemble des vecteurs $\chi_{(m)}$, m=1,2,3 qui sous-tend l'espace tangent à l'orbite du groupe, c'est-à-dire tels que $\left[\chi_{(n)}, \xi_{(m)}\right] = 0$, (m,n)=1...r. Cette dernière égalité nous indique qu'ils constituent donc une base invariante dont les constantes de structure D^l_{mn} sont introduites au moyen des commutateurs $\left[\chi_{(m)}, \chi_{(n)}\right] = D^l_{mn}\chi_{(l)}$. Afin de construire la base invariante, on se donne r vecteurs indépendants $\chi_{(n)}$ en un point P_0 avec les conditions initiales $\chi_{(n)0} = \xi_{(n)}(P_0)$, r étant le nombre de paramètres du groupe d'isométries, puis on les translate au moyen de la dérivée de Lie afin de définir r champs de vecteurs sur la variété M sur laquelle le groupe G_r agit. Si C^l_{mn} désigne les constantes de structures des vecteurs de Killing, on trouve que $D^l_{mn} = -C^l_{mn} \forall P \in M$ et donc $\left[\chi_{(m)}, \chi_{(n)}\right] = -C^l_{mn}\chi_{(l)}$. On en déduit que *l'algèbre de Lie des champs de vecteurs tangents à l'orbite, vecteurs invariants de groupe, est algébriquement équivalente à l'algèbre de Lie des vecteurs de Killing du groupe G_r*. On peut alors montrer que le produit scalaire de deux champs de vecteurs invariants quelconques est constant sur chaque orbite soit $(\chi^\alpha_{(m)}\chi^\beta_{(n)})_{;\gamma}\xi^\gamma = 0$ quelque soit le vecteur de Killing.
Par conséquent, la base invariante $(\chi_{(m)})$, construite en un point de chaque surface homogène devient un champ de vecteurs sur l'espace temps, en translatant les vecteurs invariants au moyen de la dérivée de Lie, par rapport au champ de vecteurs $n_\alpha = (-1,0,0,0)$, orthogonal aux hypersurfaces $S(t)$

$$\left[\chi_{(\mu)}, n\right] = 0 \Leftrightarrow \frac{\partial}{\partial t}(\chi^a_{(\mu)}) = 0$$

Il s'ensuit que les vecteurs invariants sont indépendants du temps et les produits scalaires $g_{ab}\chi^a_{(m)}\chi^b_{(n)}$, notés g_{mn}, sont constants sur chaque surface de transitivité et dépendent uniquement du temps. On peut désormais écrire explicitement la formulation de la métrique des modèles cosmologiques homogènes de Bianchi. Pour cela, on choisit les $\chi^{(m)}_a$ tels que $\chi^{(m)}_a\chi^a_{(n)} = \delta^m_n$. La métrique spatiotemporelle s'écrit alors sous la forme

$$ds^2 = -dt^2 + g_{mn}(t)\chi^{(m)}_a\chi^{(n)}_b dx^a dx^b$$

On définit une ***1-forme*** comme étant un opérateur linéaire agissant sur les champs de vecteurs. Ainsi, si ω est une 1-forme et $\vec{U}$ un vecteur, $\omega(\vec{U})$ est une fonction telle que $\omega(\vec{U})(P)$ est un réel, P étant un point. On introduit alors les 1-formes $(\omega^{(m)})$ telles que:

$$\omega^{(m)}_a\chi^a_{(n)} = \delta^m_n \tag{1.1}$$

On dit que les 1-formes $(\omega^{(m)})$ constituent ***la base duale*** des $(\chi_{(m)})$. Alors la matrice inverse $||\ \chi^{(m)}_a\ ||$, (m) en haut étant un indice de ligne, peut s'interpréter comme fournissant les composantes covariantes des

1-formes $\omega^{(m)}$. Les 1-formes de base vérifient les équations de Cartan et si l'on écrit

$$\omega^{(m)} = \chi_a^{(m)} dx^a \tag{1.2}$$

la forme finale de la métrique peut donc s'exprimer comme:

$$ds^2 = -dt^2 + g_{mn}(t)\omega^m\omega^n \tag{1.3}$$

1.2.3 Vecteurs invariants et métriques des modèles de Bianchi

Les vecteurs $\chi_{(n)}$ étant invariants de groupe et donc commutant avec les vecteurs de Killing, on a en termes de composantes:

$$\xi^a_{(m)}\chi^b_{(n),a} - \chi^a_{(n)}\xi^b_{(m),a} = 0 \tag{1.4}$$

Comme le déterminant de $\|\ \chi^a_{(m)}\ \|$ n'est pas nul, (m) en bas étant un indice de colonne, on peut définir trois vecteurs covariants, $\chi^{(m)}$, de composantes $\chi_a^{(m)}$ telles que $\chi^a_{(m)}\chi_b^{(m)} = \delta^a_b$. De plus, on sait que $\xi^a_{(m)}\xi_b^{(m)} = \delta^a_b$ et en reportant cette expression dans (1.4), il vient:

$$\xi^b_{(n),c} - \chi^a_{(n)}\xi^b_{(m),a}\xi_c^{(m)} = 0$$

L'équation que nous utiliserons lors du calcul des vecteurs invariants est donc:

$$\xi^a_{(n),b} - \xi^a_{(m),c}\xi_b^{(m)}\chi^c_{(n)} = 0 \tag{1.5}$$

avec pour conditions initiales de ce système différentiel en un point de coordonnées spatiales $(0{,}0{,}0)$, $\chi^a_{(m)}(0) = \xi^a_{(m)}(0)$. De plus, les vecteurs de Killing $\xi_{(m)}$ du groupe G_3 d'homogénéité spatiale correspondant aux diverses types de Bianchi ayant C^l_{mn} pour constantes de structure, vérifient

$$\xi^a_{(m)}\xi^b_{(n),a} - \xi^a_{(n)}\xi^b_{(m),a} = C^l_{mn}\xi^b_{(l)} \tag{1.6}$$

le produit de Lie de deux vecteurs de Killing étant un vecteur de Killing.

1.3 Exemple: le modèle de Bianchi de type II

Concrètement, la marche à suivre pour obtenir les bases invariantes des modèles cosmologiques de Bianchi est la suivante:

1. On suppose les constantes de structure du modèle considéré connues
2. On résout (1.6) afin d'obtenir les vecteurs de Killing $\xi^a_{(m)}$
3. On résout (1.5) afin d'obtenir les vecteurs de base invariants $\chi^a_{(m)}$
4. On écrit explicitement la métriques à l'aide de (1.1-1.3)

Les constantes de structure de chaque modèle de Bianchi figurent dans le tableau 1.2.

Ainsi, pour le modèle de Bianchi de type II, les seules constantes de structure non nulles sont $C^1_{23} = -C^1_{23} = 1$. L'équation (1.6) donne:

$$\begin{aligned}
\xi^a_{(1)}\xi^b_{(3),a} - \xi^a_{(3)}\xi^b_{(1),a} &= 0\\
\xi^a_{(1)}\xi^b_{(2),a} - \xi^a_{(2)}\xi^b_{(1),a} &= 0\\
\xi^a_{(2)}\xi^b_{(3),a} - \xi^a_{(3)}\xi^b_{(2),a} &= 0
\end{aligned}$$

dont une solution particulière est:

$$\|\ \xi^a_{(m)}\ \| = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & x^3 \\ 0 & 1 & 0 \end{pmatrix} \text{ et } \|\ \xi^a_{(m)}\ \|^{-1} = \|\ \xi^{(m)_a}\ \| = \begin{pmatrix} -x^3 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

le (m) en bas (en haut) étant un indice de colonne(respectivement de ligne). L'équation (1.5) va alors s'écrire

$\chi^1_{(n),b} = 0 \Rightarrow \chi^1_{(n)}$ est constant pour tout n.

$\chi^3_{(n),b} = 0 \Rightarrow \chi^3_{(n)}$ est constant pour tout n.

$\chi^2_{(n),b} = \xi^{(3)}\chi^3_{(n)}$ où $\xi_b^{(3)}$est nul sauf lorsque b=1, auquel cas $\xi_1^{(3)} = 1$

Classe A	Constantes de structure
I	$C^{\lambda}_{\mu\nu} = 0$
II	$C^1_{23} = -C^1_{32} = 1$
VI_0	$C^1_{23} = -C^1_{32} = 1$, $C^2_{13} = -C^2_{31} = 1$
VII_0	$C^1_{23} = -C^1_{32} = 1$, $C^2_{13} = -C^2_{31} = -1$
$VIII$	$C^1_{23} = -C^1_{32} = 1$, $C^2_{31} = -C^2_{13} = 1$, $C^3_{12} = -C^3_{21} = -1$
IX	$C^1_{23} = -C^1_{32} = 1$, $C^2_{31} = -C^2_{13} = 1$, $C^3_{12} = -C^3_{21} = 1$
Classe B	**Constantes de structure**
V	$C^1_{13} = -C^1_{31} = -1$, $C^2_{23} = -C^2_{32} = -1$
IV	$C^1_{13} = -C^1_{31} = -1$,$C^1_{23} = -C^1_{32} = 1$,$C^2_{23} = -C^2_{32} = -1$
VI_h	$C^1_{23} = -C^1_{32} = 1$, $C^2_{13} = -C^2_{31} = 1$, $C^1_{13} = -C^1_{31} = -\sqrt{-h}$, $C^2_{23} = -C^2_{32} = -\sqrt{-h}$
VII_h	$C^1_{23} = -C^1_{32} = 1$, $C^2_{13} = -C^2_{31} = -1$, $C^1_{13} = -C^1_{31} = -\sqrt{h}$, $C^2_{23} = -C^2_{32} = -\sqrt{h}$

TAB. 1.2 – *Les constantes de structure des modèles de Bianchi*

Cette dernière équation donne

$$\left.\begin{array}{c} \chi^2_{(n),1} = \chi^3_{(n)} \\ \chi^2_{(n),2} = 0 \\ \chi^2_{(n),3} = 0 \end{array}\right\} \Rightarrow \chi^2_{(n)} = \chi^3_{(n)} x^1 + const \text{ pour tout n}$$

Partant de ces solutions, on forme trois vecteurs invariants de base:

$$\| \chi^a_{(n)} \| = \begin{pmatrix} 0 & 0 & 1 \\ 1 & x^1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ et } \| \chi^a_{(n)} \|^{-1} = \| \chi^{(n)}_a \| = \begin{pmatrix} 0 & 1 & -x^1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

L'équation (1.2) permet d'écrire

$$\begin{aligned} \omega^1 &= dx^2 - x^1 dx^3 \\ \omega^2 &= dx^3 \\ \omega^3 &= dx^1 \end{aligned}$$

d'où la métrique diagonale de type II de Bianchi

$$ds^2 = -dt^2 + g_{11}(t)(dx^2 - x^2 dx^3)^2 + g_{22}(t)(dx^3)^2 + g_{33}(t)(dx^1)^2$$

Dans le tableau 1.3, les 1-formes définissant chaque type de Bianchi sont indiquées.

Classe A	ω^1	ω^2	ω^3
I	dx^1	dx^2	dx^3
II	$dx^2 - x^1 dx^3$	dx^3	dx^1
VI_0	$chx^1 dx^2 + shx^1 dx^3$	$shx^1 dx^2 + chx^1 dx^3$	$-dx^1$
VII_0	$cosx^1 dx^2 + sinx^1 dx^3$	$-sinx^1 dx^2 + cosx^1 dx^3$	$-dx^1$
$VIII$	$dx^1 + ((x^1)^2 - 1)dx^2 +$ $(x^1 + x^2 - (x^1)^2 x^2)dx^3$	$2dx^1 dx^2 + (1 - 2x^1 x^2)dx^3$	$-dx^1 - (1 + (x^1)^2)dx^2 +$ $(x^2 - x^1 + (x^1)^2 x^2)dx^3$
IX	$-sinx^3 dx^1 + sinx^1 cosx^3 dx^2$	$cosx^3 dx^1 + sinx^1 sinx^3 dx^2$	$cosx^1 dx^2 + dx^3$
Classe B			
V	$e^{-x^1} dx^2$	$e^{-x^1} dx^3$	$-dx^1$
IV	$e^{-x^1} dx^2 + x^1 e^{-x^1} dx^3$	$e^{x^1} dx^3$	$-dx^1$
VI_h	$e^{-ax^1} chx^1 dx^2 +$ $e^{-ax^1} shx^1 dx^3$	$e^{-ax^1} shx^1 dx^2 +$ $e^{-ax^1} chx^1 dx^3$	$-dx^1$
VII_h	$e^{-ax^1} cosx^1 dx^2 +$ $e^{-ax^1} sinx^1 dx^3$	$-e^{-ax^1} sinx^1 dx^2 +$ $e^{-ax^1} cosx^1 dx^3$	$-dx^1$

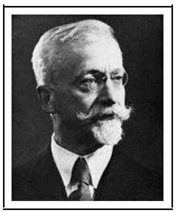
TAB. 1.3 – *Les 1-formes définissant les métriques diagonales de Bianchi*

Chapitre 2

Ecriture des équations de champs des théories tenseur-scalaires

Ce chapitre se décompose en trois sections. Dans la première on montre comment obtenir rapidement les composantes non nulles du tenseur de courbure par la méthode de Cartan. Dans la seconde, partant de la forme la plus générale de Lagrangien pour les théories tenseur-scalaires, les équations de champs des modèles de Bianchi de la classe A sont déduites. Enfin dans la troisième, on étudie le formalisme Hamiltonien afin de trouver la forme des contraintes Hamiltoniennes dont nous nous servirons plus tard pour obtenir les équations de Hamilton.

2.1 Calcul de la courbure d'une variété par la méthode de Cartan



Avant de nous lancer dans les calculs, commençons par une petite biograpie de Cartan dont le nom va revenir tout au long de cette section. Les sources de cette biographie que nous reproduisons ici, se trouve sur le web à l'adresse http://www.iecn.u-nancy.fr/Les-Maths-A-Nancy/.
Élie Cartan est né le 9 avril 1869 à Dolomieu (Dauphiné) où il fit ses études primaires. Son père était le forgeron du village. Il poursuivi ses études au collège de Vienne puis au lycée de Grenoble. Au lycée Jeanson-de-Sailly, il suit la préparation à l'Ecole Normale Supérieure où il entre en 1888. Ses enseignants se nomment alors H. Poincaré, E. Picard et de C. Hermite. A la suite de ces études, il obtient une bourse de la fondation Peccot. Ses premiers travaux, qui débouchèrent sur sa thèse soutenue en 1894, portent sur les groupes de Lie simples complexes, où il reprend, corrige et développe les résultats de structure et de classification obtenus par W. Killing.
Il commence alors sa carrière en obtenant un poste de lecteur à l'Université de Montpellier de 1894 à 1896, puis à la Faculté des sciences de Lyon de 1896 à 1903. En 1903, il est nommé professeur à la Faculté des sciences de Nancy, où il restera jusqu'en 1909. Il donne en même temps des cours à l'Ecole d'Electrotechnique et de Mécanique Appliquée. Il rédige deux grands articles sur une généralisation en dimension infinie des groupes de Lie simples et il élabore entre autre la théorie des formes extérieures dont nous allons découvrir quelques éléments dans ce qui suit.
En 1909, il vient enseigner à la Sorbonne, où il est nommé professeur en 1912. Il assure par ailleurs un enseignement à l'Ecole de Physique et Chimie de Paris. En 1914, il résout le problème de la classification des groupes de Lie simples réels et détermine les représentations de dimension finie de ces groupes. Pendant la guerre, il sert comme sergent dans l'hôpital aménagé dans les locaux de l'Ecole Normale Supérieure, tout en continuant ses travaux en mathématiques. Son oeuvre mathématique ultérieure est considérable, avec près de 200 publications et de nombreux ouvrages. Parmi les thèmes abordés, mentionnons l'étude des variétés à courbure constante négative, la théorie de la gravitation d'Einstein, la théorie des connexions affines, les groupes d'holonomie, les espaces riemanniens symétriques, les spineurs. Il est aussi l'auteur de plusieurs articles sur l'histoire de la géométrie.
Il prit sa retraite en 1940, et mourut le 6 mai 1951.

2.1.1 Différentiation des 1-formes de base

Nous allons établir les équations de structure de Cartan qui permettent de trouver la courbure d'une variété sans avoir à calculer les composantes nulle du tenseur de courbure. A cette fin, introduisons le

concept de ***différentiation des 1-formes de base***.
Soit $\{\vec{e}_i\}$, une base de vecteurs d'un espace de Riemann et $\{\tilde{\omega}_i\}$ une base de 1-forme duale de la base des $\{\vec{e}_i\}$. On a:

$$\begin{aligned} \vec{e}_i &= a_i^s \partial/\partial x^s \\ \tilde{\omega}_i &= b_s^i d\tilde{x}^s \end{aligned}$$

a_s et b_s étant des fonctions du temps t et donc, du fait de la relation de dualité:

$$< \tilde{\omega}_i, \vec{e}_i >= b_s^i a_j^t \delta_t^s = \delta_j^i$$

soit

$$b_s^i a_j^s = \delta_j^i \tag{2.1}$$

On définit le ***produit extérieur*** d'une 1-forme par une 1-forme de la manière suivante:

$$\begin{aligned} \tilde{\mu} \wedge \tilde{\nu} &= \tilde{\mu} \otimes \tilde{\nu} - \tilde{\nu} \otimes \tilde{\mu} \\ \tilde{\mu} \wedge \tilde{\nu} &= -\tilde{\nu} \wedge \tilde{\mu} \\ \tilde{\mu} \wedge \tilde{\mu} &= 0 \end{aligned}$$

où $\otimes$ désigne le produit tensoriel. Alors la ***différentielle extérieure*** d'une 1-forme de base s'écrira:

$$\tilde{d}\tilde{\omega}^i = \tilde{d}b_s^i \wedge \tilde{d}x^s = b_{s,t}^i \tilde{d}x^t \wedge \tilde{d}x^s$$

car $\tilde{d}(\tilde{d}x^s) = 0$. A l'aide de (2.1), on obtient alors:

$$\tilde{d}\tilde{\omega}^i = b_{s,t}^i a_j^t a_k^s \tilde{\omega}^j \wedge \tilde{\omega}^k \tag{2.2}$$

et

$$(b_s^i a_k^s)_{,t} = (\delta_k^i)_{,t} = 0 \Rightarrow b_{s,t}^i a_k^s = -b_s^i a_{k,t}^s$$

Par conséquent, il vient pour (2.2):

$$\tilde{d}\tilde{\omega}^i = -b_s^i a_{k,t}^s a_j^t \tilde{\omega}^j \wedge \tilde{\omega}^k \tag{2.3}$$

Or seule la partie antisymétrique du coefficient de $\tilde{\omega}^j \wedge \tilde{\omega}^k$ importe car cette expression est antisymétrique sur les indices j et k, d'où:

$$\tilde{d}\tilde{\omega}^i = -\frac{1}{2} b_s^i (a_j^t a_{k,t}^s - a_k^t a_{j,t}^s) \tilde{\omega}^j \wedge \tilde{\omega}^k$$

De plus, on sait que:

$$[\vec{e}_j, \vec{e}_k] = (a_j^t a_{k,t}^s - a_k^t a_{j,t}^s) b_s^i \vec{e}_i = C_{jk}^i \vec{e}_i$$

le commutateur de deux vecteurs de base étant encore un vecteur de l'espace vectoriel de base et les C_{jk}^i étant les coefficients de structure de la base considérée. D'où:

$$\tilde{d}\tilde{\omega}^i = -\frac{1}{2} C_{jk}^i \tilde{\omega}^j \wedge \tilde{\omega}^k \tag{2.4}$$

Cette équation donne la ***différentielle extérieure des 1-formes de base*** en termes du produit extérieur de ces 1-formes de base.

2.1.2 Les équations de structure de Cartan

On définit l'ensemble des 1-formes de connexion affine $\tilde{\omega}_j^i$ par:

$$\tilde{\omega}_j^i = \Gamma_{jk}^i \tilde{\omega}^k$$

avec $\nabla_i \vec{e}_j = \Gamma_{ji}^k \vec{e}_k$, ∇ étant la connexion affine de composantes Γ et avec la notation $\nabla_i = \nabla_{\vec{e}_i}$ pour tout vecteur $\vec{e}_i$ appartenant à la base définie plus haut. Nous ne considèrerons exclusivement que des connexions affines symétriques, c'est-à-dire telles que:

$$\nabla_{\vec{u}} \vec{v} - \nabla_{\vec{v}} \vec{u} = [\vec{u}, \vec{v}]$$

quels que soient les champs de vecteurs $\vec{u}$ et $\vec{v}$. Cette condition de symétrie permet de déduire pour les vecteurs de base:

$$\Gamma^i_{kj} - \Gamma^i_{jk} = C^i_{jk}$$

Ainsi, (2.4) devient:

$$\tilde{d}\tilde{\omega}^i = -\Gamma^i_{kj}\tilde{\omega}^j \wedge \tilde{\omega}^k$$

donnant ***la première équation de structure de Cartan***:

$$\tilde{d}\tilde{\omega}^i = -\tilde{\omega}^i_k \wedge \tilde{\omega}^k \tag{2.5}$$

Afin d'obtenir la seconde équation de structure de Cartan, il nous faut calculer la différentielle extérieure des 1-formes de connexion affine :

$$\tilde{d}\tilde{\omega}^i_j = \Gamma^i_{js,t}\tilde{\omega}^t \wedge \tilde{\omega}^s - \frac{1}{2}\Gamma^i_{jl}\tilde{\omega}^t \wedge \tilde{\omega}^s \tag{2.6}$$

De plus:

$$\tilde{\omega}^i_l \wedge \tilde{\omega}^l_j = \Gamma^i_{lt}\Gamma^l_{js}\tilde{\omega}^t \wedge \tilde{\omega}^s \tag{2.7}$$

Seule la partie antisymétrique des coefficients de $\tilde{\omega}^t \wedge \tilde{\omega}^s$ importe dans les relations (2.6-2.7). En les sommant, il vient:

$$\tilde{d}\tilde{\omega}^i_j + \tilde{\omega}^i_s \wedge \tilde{\omega}^s_j = \frac{1}{2}(\Gamma^i_{js,t} - \Gamma^i_{jt,s} - \Gamma^i_{jl}C^l_{ts} + \Gamma^i_{lt}\Gamma^l_{js} - \Gamma^i_{ls}\Gamma^l_{jt})\tilde{\omega}^t \wedge \tilde{\omega}^s$$

Or l'opérateur de courbure est défini par:

$$\begin{aligned} R(\vec{e}_s,\vec{e}_t)\vec{e}_j &= \bigtriangledown_s(\bigtriangledown_t\vec{e}_j) - \bigtriangledown_t(\bigtriangledown_s\vec{e}_j) - \bigtriangledown_{[\vec{e}_s,\vec{e}_t]}\vec{e}_j \\ &= \bigtriangledown_s(\bigtriangledown_t\vec{e}_j) - \bigtriangledown_t(\bigtriangledown_s\vec{e}_j) - \bigtriangledown_{[\vec{e}_s,\vec{e}_t]}\vec{e}_j - C^l_{st}\bigtriangledown_l \vec{e}_j \\ &= R^i_{jst}\vec{e}_i \end{aligned}$$

car $[\vec{e}_s,\vec{e}_t] = C^l_{st}\vec{e}_l$ et ou R^i_{jst} désigne les composantes du tenseur de Riemann. D'où:

$$R^i_{jst} = -\Gamma^i_{js,t} + \Gamma^i_{jt,s} + \Gamma^i_{jl}C^l_{ts} - \Gamma^i_{lt}\Gamma^l_{js} + \Gamma^i_{ls}\Gamma^l_{jt}$$

On obtient alors ***la seconde équation de structure de Cartan***:

$$\tilde{d}\tilde{\omega}^i_j + \tilde{\omega}^i_s \wedge \tilde{\omega}^s_j = \frac{1}{2}R^i_{jst}\omega^s \wedge \tilde{\omega}^t \tag{2.8}$$

Ce sont ces deux équations de strucuture de Cartan qui servent au calcul du tenseur de courbure comme nous allons l'expliquer dans les sections suivantes.

2.1.3 La méthode de Cartan

Soit une variété M et sa métrique riemannienne g, (M,g) donne univoquement naissance à une dérivation covariante symétrique $\bigtriangledown$ associée. La ***condition de compatibilité riemannienne*** écrite ci-dessous garantit la compatibilité de $\bigtriangledown$ et g

$$\bigtriangledown_{\vec{w}}(g(\vec{u},\vec{v})) = g(\bigtriangledown_{\vec{w}}\vec{u},\vec{v}) + g(\vec{u},\bigtriangledown_{\vec{w}}\vec{v})$$

où $\vec{u}$, $\vec{v}$ et $\vec{w}$ sont des champs de vecteurs et g le tenseur métrique de composantes g_{ij}. Cette condition associée à la première équation de structure de Cartan permet de calculer univoquement les formes et symboles de connexions à partir de la métrique et de la différentielle des formes de base. Elle s'écrit alors:

$$\tilde{d}g_{ij} = \tilde{\omega}_{ij} + \tilde{\omega}_{ji}$$

avec $\tilde{\omega}_{ij} = g_{is}\tilde{\omega}^s_j = \Gamma_{ijk}\tilde{\omega}^k$. Ainsi, si on choisit une base telle que $g_{ij} = const$, il vient:

$$\tilde{d}g_{ij} = 0 \text{ et } \tilde{\omega}_{ij} = -\tilde{\omega}_{ji} \tag{2.9}$$

La ***méthode de Cartan*** est alors:

- On choisit une tétrade de vecteurs de base $\{\vec{e}_i\}$ et la tétrade duale de $1-formes$ de base $\{\tilde{\omega}^i\}$ correspondante telles que $g_{ij} = \vec{e}_i.\vec{e}_j = const$ afin de pouvoir utiliser (2.9).
- On résout la première équation de Cartan en utilisant (2.9) et on obtient alors les six 1-formes de connexion affine $\tilde{\omega}^i_j$.
- On utilise ces 2-formes afin de calculer les six 2-formes de courbure $\tilde{\theta}^i_j = \tilde{d}\tilde{\omega}^i_j + \tilde{\omega}^i_j \wedge \tilde{\omega}^l_j$ et la deuxième équation de structure afin d'obtenir les composantes du tenseur de courbure de Riemann dans la base choisie.

2.1.4 Application de la méthode de Cartan

Nous allons montrer les premiers pas du calcul dans la base de Cartan canonique [1] des formes différentielles $\{\tilde{\omega}^i\}$ invariantes sous SO(3) et qui engendrent l'espace homogène de type IX de Bianchi, des composantes du tenseur de courbure de Riemann en appliquant la méthode de Cartan. Pour cela, nous écrivons la métrique sous la forme:

$$ds^2 = -dt^2 + e^{2\alpha}(\tilde{\omega}^1)^2 + e^{2\beta}(\tilde{\omega}^2)^2 + e^{2\gamma}(\tilde{\omega}^3)^2$$

où les fonctions α, β et γ ne dépendent que de t et les ω^i sont les 1-formes définisant l'espace homogène de Bianchi de type IX:

$$\begin{aligned}
\tilde{\omega}_0 &= \tilde{d}t \\
\tilde{\omega}_1 &= -sin(z)\tilde{d}x + sin(x)cos(z)\tilde{d}y \\
\tilde{\omega}_2 &= cos(z)\tilde{d}x + sin(x)sin(z)\tilde{d}y \\
\tilde{\omega}_3 &= cos(x)\tilde{d}y + \tilde{d}z
\end{aligned}$$

On calcule alors que:

$$\begin{aligned}
\tilde{d}\tilde{\omega}_0 &= 0 \\
\tilde{d}\tilde{\omega}_1 &= -cos(z)\tilde{d}z \wedge \tilde{d}x + cos(x)cos(z)\tilde{d}x \wedge \tilde{d}y - sin(x)sin(z)\tilde{d}z \wedge \tilde{d}y \\
\tilde{d}\tilde{\omega}_2 &= -sin(z)\tilde{d}z \wedge \tilde{d}x + cos(x)sin(z)\tilde{d}x \wedge \tilde{d}y - sin(x)cos(z)\tilde{d}z \wedge \tilde{d}y \\
\tilde{d}\tilde{\omega}_3 &= -sin(x)\tilde{d}x \wedge \tilde{d}y
\end{aligned}$$

d'où

$$\begin{aligned}
\tilde{d}\tilde{\omega}^1 &= \tilde{\omega}^2 \wedge \tilde{\omega}^3 \\
\tilde{d}\tilde{\omega}^2 &= \tilde{\omega}^3 \wedge \tilde{\omega}^1 \\
\tilde{d}\tilde{\omega}^3 &= \tilde{\omega}^1 \wedge \tilde{\omega}^2
\end{aligned}$$

On choisit une nouvelle base de 1-forme telle que les fonctions métriques g_{ij} soient des constantes et une nouvelle coordonnée temporelle τ:

$$\begin{aligned}
\tilde{\nu}^0 &= \tilde{d}t = e^{\alpha+\beta+\gamma}\tilde{d}\tau \\
\tilde{\nu}^i &= e^{\alpha_i}\tilde{\omega}^i
\end{aligned}$$

avec $i = 1{,}2{,}3$ et $\alpha_i = \alpha{,}\beta{,}\gamma$ et pas de sommation sur i. La métrique s'écrit alors:

$$ds^2 = -(\tilde{\nu}^0)^2 + (\tilde{\nu}^1)^2 + (\tilde{\nu}^2)^2 + (\tilde{\nu}^3)^2$$

et on calcule que:

$$\begin{aligned}
\tilde{d}\tilde{\nu}^1 = \alpha' e^{-\alpha-\beta-\gamma}\tilde{\nu}^0 \wedge \tilde{\nu}^1 + e^{\alpha-\beta-\gamma}\tilde{\nu}^2 \wedge \tilde{\nu}^3 \\
\tilde{d}\tilde{\nu}^2 = \beta' e^{-\alpha-\beta-\gamma}\tilde{\nu}^0 \wedge \tilde{\nu}^2 + e^{\beta-\alpha-\gamma}\tilde{\nu}^3 \wedge \tilde{\nu}^1 \\
\tilde{d}\tilde{\nu}^3 = \gamma' e^{-\alpha-\beta-\gamma}\tilde{\nu}^0 \wedge \tilde{\nu}^3 + e^{\gamma-\alpha-\beta}\tilde{\nu}^1 \wedge \tilde{\nu}^2
\end{aligned}$$

On se sert maintenant de la première équation de Cartan sachant que par antisymétrie:

$$\begin{aligned}
\tilde{\nu}^\eta_\eta &= 0 \\
\tilde{\nu}^0_\eta &= \tilde{\nu}^\eta_0 \\
\tilde{\nu}^n_m &= \tilde{\nu}^m_n
\end{aligned}$$

1. La base de Cartan canonique est la base o`u un maximum de constantes de structure valent 0 ou ± 1

sans sommation sur $\eta = 0{,}1{,}2{,}3$. La première équation de Cartan et le calcul des $\tilde{d}\tilde{\nu}^i$ ci-dessus permettent d'écrire:

$$\begin{aligned}
-\tilde{d}\tilde{\nu}^0 &= \tilde{\nu}^0_1 \wedge \tilde{\nu}^1 + \tilde{\nu}^0_2 \wedge \tilde{\nu}^2 + \tilde{\nu}^0_3 \wedge \tilde{\nu}^3 = 0 \\
-\tilde{d}\tilde{\nu}^1 &= \tilde{\nu}^1_0 \wedge \tilde{\nu}^1 + \tilde{\nu}^1_2 \wedge \tilde{\nu}^2 + \tilde{\nu}^1_3 \wedge \tilde{\nu}^3 = -(\alpha' e^{-\alpha-\beta-\gamma}\tilde{\nu}^0 \wedge \tilde{\nu}^1 + e^{\alpha-\beta-\gamma}\tilde{\nu}^2 \wedge \tilde{\nu}^3) \\
-\tilde{d}\tilde{\nu}^2 &= \tilde{\nu}^2_0 \wedge \tilde{\nu}^0 + \tilde{\nu}^2_1 \wedge \tilde{\nu}^1 + \tilde{\nu}^2_3 \wedge \tilde{\nu}^3 = -(\beta' e^{-\alpha-\beta-\gamma}\tilde{\nu}^0 \wedge \tilde{\nu}^2 + e^{\beta-\alpha-\gamma}\tilde{\nu}^3 \wedge \tilde{\nu}^1) \\
-\tilde{d}\tilde{\nu}^3 &= \tilde{\nu}^3_0 \wedge \tilde{\nu}^0 + \tilde{\nu}^3_1 \wedge \tilde{\nu}^1 + \tilde{\nu}^3_2 \wedge \tilde{\nu}^2 = -(\gamma' e^{-\alpha-\beta-\gamma}\tilde{\nu}^0 \wedge \tilde{\nu}^3 + e^{\gamma-\alpha-\beta}\tilde{\nu}^1 \wedge \tilde{\nu}^2)
\end{aligned}$$

En examinant ce système d'équation, il vient que:

$$\begin{aligned}
\tilde{\nu}^1_0 &= \alpha' e^{-\alpha-\beta-\gamma}\tilde{\nu}^1 \\
\tilde{\nu}^2_0 &= \beta' e^{-\alpha-\beta-\gamma}\tilde{\nu}^2 \\
\tilde{\nu}^3_0 &= \gamma' e^{-\alpha-\beta-\gamma}\tilde{\nu}^3 \\
\tilde{\nu}^1_2 &= \frac{1}{2}e^{-\alpha-\beta-\gamma}(e^{2\alpha} + e^{2\beta} - e^{2\gamma})\tilde{\nu}^3 \\
\tilde{\nu}^3_1 &= \frac{1}{2}e^{-\alpha-\beta-\gamma}(e^{2\alpha} - e^{2\beta} + e^{2\gamma})\tilde{\nu}^2 \\
\tilde{\nu}^2_3 &= \frac{1}{2}e^{-\alpha-\beta-\gamma}(-e^{2\alpha} + e^{2\beta} + e^{2\gamma})\tilde{\nu}^1
\end{aligned}$$

Afin de se servir de la deuxième équation de Cartan, on calcule les différentielles extérieures des $\tilde{\nu}^i_j$ ainsi que leur produits extérieurs dont on extrait les 2 formes de courbure:

$$\tilde{\theta}^u_v = \tilde{d}\tilde{\nu}^u_v + \tilde{\nu}^u_s \wedge \tilde{\nu}^s_v = \frac{1}{2}R^u_{vst}\tilde{\nu}^s \wedge \tilde{\nu}^t \text{ avec u, v, s et t variant de 0 à 3.}$$

Par identification, on obtient donc les composantes du tenseur de Riemann. Dans ce qui suit, nous ommetrons les tildes sur les 1-formes afin d'alléger l'écriture.

2.2 Le formalisme Lagrangien

Sachant désormais calculer le tenseur de courbure d'un espace homogène, nous allons voir comment établir les équations de champs d'une théorie tenseur-scalaire spécifiée par un Lagrangien.

2.2.1 Forme Générale des équations de champs

L'action d'une théorie tenseur-scalaire peut être écrite de la manière suivante:

$$S = \int \left[G^{-1}R - \frac{1}{2}\frac{2\omega+3}{\phi^2}\phi_{,\mu}\phi^{,\mu} - U + 16\pi L_m\right]\sqrt{-g}$$

L_m représente le Lagrangien d'un fluide parfait d'équation d'état $p = (\delta - 1)\rho$, où p et ρ sont respectivement la pression et la densité du fluide. G, ω et U sont trois fonctions du champ scalaire ϕ dont nous allons commenter la signification.

- G est la fonction de gravitation. Lorsqu'elle est une constante, on dit que le ***champ scalaire est minimalement couplé***.
- ω est la ***fonction de couplage de Brans-Dicke***. Elle représente le couplage du champ scalaire avec la métrique et est ainsi appelée car lorsqu'elle vaut une constante, on retrouve le couplage de la théorie de Jordan, Brans et Dicke.
- U est le potentiel et représente le couplage du champ avec lui même. Lorsque $U \neq 0$, on dit que le ***champ scalaire est massif***.

Cette action n'est pas la plus générale qu'il soit pour une théorie tenseur-scalaire mais est représentative de la plupart des théories étudiées dans la littérature. Ainsi:

- La théorie de la Relativité Générale avec une constante cosmologique et un fluide parfait, souvent considérée comme le modèle capable de décrire notre Univers actuel, est tel que G représente la constante de gravitation, ω n'apparaît pas dans l'action et $U = 2\Lambda$, Λ étant la constante cosmologique.

- La théorie de Brans-Dicke est retrouvée pour $G = \phi^{-1}$ et $\omega = const$. Cette théorie a été initialement imaginée pour obtenir une théorie relativiste de la gravitation compatible avec les idées de Mach et est telle que la fonction de gravitation varie comme l'inverse du champ scalaire.
- La théorie des cordes à basse énergie sans son tenseur antisymétrique est définie par $G = e^{\phi}$ et $3 + 2\omega = \phi e^{-\phi}$

En variant l'action par rapport aux fonctions métriques, on obtient les équations de champs:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = G[\frac{1}{2}\frac{2\omega+3}{\phi^2}\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}\frac{2\omega+3}{2\phi^2}\phi_{,\lambda}\phi^{\lambda}g_{\mu\nu} + (G^{-1})_{,\mu;\nu} - g_{\mu\nu}\Box(G^{-1}) - \frac{1}{2}Ug_{\mu\nu} + \frac{8\pi}{c^4}T_{\mu\nu}]$$

La courbure scalaire R vaut:

$$R = G\left[\frac{1}{2}\frac{2\omega+3}{\phi^2}\phi_{\lambda}\phi^{\lambda} + 3\Box G^{-1} + 2U - \frac{8\pi}{c^4}T\right]$$

On en déduit une forme alternative des équations de champs:

$$R_{\mu\nu} = G\left[\frac{1}{2}\frac{2\omega+3}{\phi^2}\phi_{,\mu}\phi_{,\nu} + G^{-1}_{,\mu;\nu} + \frac{1}{2}g_{\mu\nu}\Box G^{-1} + \frac{1}{2}g_{\mu\nu}U + \frac{8\pi}{c^4}(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)\right] \tag{2.10}$$

L'équation de Klein-Gordon est obtenue en variant l'action par rapport au champ scalaire, ce qui nous donne:

$$GG^{-1}_{\phi}(\frac{1}{2}\frac{2\omega+3}{\phi^2}\phi_{,\lambda}\phi^{,\lambda} + 3\Box G^{-1} + 2U - \frac{8\pi}{c^4}T) + (-\frac{\omega_{\phi}}{\phi^2} - \frac{3+2\omega}{\phi^2})\phi_{,\lambda}\phi^{\lambda} - U_{\phi} + \frac{2\omega+3}{\phi^2}\Box\phi = 0 \tag{2.11}$$

2.2.2 Equations de champs pour les modèles de Bianchi de la classe A

Introduisant les différentes métriques des modèles de Bianchi de la classe A qui s'écrivent sous la forme

$$ds^2 = e^{2\alpha+2\beta+2\gamma}d\tau^2 + e^{2\alpha}(\omega^1)^2 + e^{2\beta}(\omega^2)^2 + e^{2\gamma}(\omega^3)^2$$

dans les équations de champs (2.10), il vient:

$$\alpha''G^{-1} + \alpha'(G^{-1})' + (1/2G^{-1})'' - e^{6\Omega}\left[1/2U + 4\pi(2-\delta)\rho_0 V^{-3\delta}\right] = G^{-1}C_{Lag_1}$$
$$\beta''G^{-1} + \beta'(G^{-1})' + 1/2(G^{-1})' - e^{6\Omega}\left[1/2U + 4\pi(2-\delta)\rho_0 V^{-3\delta}\right] = G^{-1}C_{Lag_2}$$
$$\gamma''G^{-1} + \gamma'(G^{-1})' + 1/2(G^{-1})'' - e^{6\Omega}\left[1/2U + 4\pi(2-\delta)\rho_0 V^{-3\delta}\right] = G^{-1}C_{Lag_3}$$

$$\alpha'\beta' + \alpha'\gamma' + \beta'\gamma' + 3(\alpha'+\beta'+\gamma')GG^{-1,} - 1/2GV^2U - \frac{1}{4}G(3+2\omega)\frac{\phi'^2}{\phi^2} - 8\pi\rho_0 GV^{2-\delta} = C_{Lag_4}$$

où V représente le 3-volume de l'Univers $e^{\alpha+\beta+\gamma}$ et ou la densité d'énergie du fluide parfait a été calculé en utilisant son équation de conservation et s'écrit:

$$\rho = V^{-3\delta}$$

Les C_{Lag_i} représentent les potentiels de courbure des différents modèles de Bianchi de classe A et sont reproduits dans le tableau 2.1.

2.3 Le formalisme hamiltonien ADM

Il existe trois formulations hamiltoniennes principales de la relativité générale: ce sont les approches d'Arnowitt, Deser et Misner (ADM), de Dirac et de Kuchař. La méthode ADM choisit de résoudre les contraintes primaires qui proviennent du lagrangien singulier de la théorie et développe ensuite la formulation hamiltonienne en utilisant seulement les variables indépendantes dans l'espace de phases. Il faut

I	$C_{Lag_i} = 0$
II	$-C_{Lag_1} = C_{Lag_2} = C_{Lag_3} = 2C_{Lag_4} = \frac{1}{2}e^{4\alpha}$
VI_0 et VII_0	$-C_{Lag_1} = +C_{Lag_2} = \frac{1}{2}(e^{4\alpha} - e^{4\beta}),\ C_{Lag_3} = 2C_{Lag_4} = \frac{1}{2}(e^{2\alpha} \pm e^{2\beta})^2$
$VIII$ et IX	$C_{lag_1} = \frac{1}{2}\left[(e^{2\beta} \pm e^{2\gamma})^2 - e^{4\alpha}\right],\ C_{lag_2} = \frac{1}{2}\left[(e^{2\alpha} \pm e^{2\gamma})^2 - e^{4\beta}\right]$ $C_{lag_3} = \frac{1}{2}\left[(e^{2\alpha} - e^{2\beta})^2 - e^{4\gamma}\right]$ $C_{lag_4} = \frac{1}{4}\left[e^{4\alpha} + e^{4\beta} + e^{4\gamma} - 2(e^{2(\alpha+\beta)} \mp e^{2(\alpha+\gamma)} \mp e^{2(\beta+\gamma)})\right]$

TAB. 2.1 – *Les potentiels de courbure des modèles de Bianchi pour le formalisme Lagrangien*

cependant souligner que dans le cadre général des théories de jauge, cette manière de procéder ne peut être considérée comme idéale: elle occulte en effet généralement la covariance vis-à-vis de symétries du groupe de Poincaré et, dans le cas de contraintes reliées à une invariance de jauge locale, elle ne parvient pas toujours à mettre en relief clairement certains aspects de l'invariance de jauge. Dans le contexte qui nous intéresse ici, la résolution ADM des contraintes simplifie grandement le formalisme (qui ne comporte plus de degrés de liberté redondants) ainsi que l'interprétation physique des résultats obtenus (ceci est particulièrement intéressant pour l'étude de l'influence de la courbure spatiale sur l'approche asymptotique de la singularité de modèles anisotropes). L'approche de Dirac découle directement de la théorie de Dirac des systèmes contraints et incorpore, sans les résoudre, les contraintes dans le formalisme; elle est particulièrement adaptée à la quantification canonique de la théorie. Quant à la formulation de Kuchař, également intéressante au niveau de la quantification, elle place l'accent sur la signification géométrique de la formulation hamiltonienne de la relativité générale. Nous suivons ici la méthode ADM.

La démonstration des résultats ADM est particulièrement laborieuse ; on ne trouve souvent pas, dans la littérature, ces calculs effectués explicitement. L'appendice B du mémoire de licence de G. Rossi (*Formalisme hamiltonien en relativité générale et en cosmologie*, Université de Liège, Faculté des sciences, Institut de mathématiques, 1973-1974), dirigé par J. Demaret, indique les principales étapes techniques. Des notes non publiées de P. Tombal et de A. Moussiaux (*Le formalisme hamiltonien en relativité générale (première version)*, Facultés universitaires Notre-Dame de la Paix, Namur) et un document également non publié de C. Scheen (*Introduction au formalisme hamiltonien ADM de la relativité générale*, Université de Liège, Institut d'astrophysique et de géophysique, 1992-1993), plus systématique et complet, présentent tous les détails de calcul. Je remercie tout particulièrement le docteur Christian Scheen qui a corrigé cette section. qui s'inspire de ces trois travaux et propose un résumé des étapes les plus techniques.

Le formalisme hamiltonien présente plusieurs avantages sur le formalisme lagrangien. Il permet d'écrire les équations de champs sous la forme d'un système du premier ordre au lieu du second et l'interprétation physique des résultats y est plus facile comme le montre par exemple la clareté de l'approche chaotique de la singularité expliquée par Misner en utilisant le formalisme ADM par rapport à celle utilisée par Belinskii-Khalatnikov-Lifshitz(BKL) avec le formalisme lagrangien.

Dans un premier temps, nous allons rechercher la forme hamiltonienne de l'action de la relativité générale:

$$S = \int_M R\sqrt{-g}\, d^4x \tag{2.12}$$

que nous généraliserons au cas de la présence d'un champ scalaire.

La relativité générale est l'exemple typique d'une théorie qui a la propriété de covariance pour tout changement de coordonnées dans l'espace-temps ; on dit encore qu'elle est paramétrisée *a priori*. En théorie hamiltonienne classique, il est possible d'inclure la variable temporelle dans les variables dynamiques ; la paramétrisation de cette théorie fait apparaître des contraintes et le problème variationnel est d'extrémaliser une forme de l'action où ces contraintes sont introduites *via* des multiplicateurs de Lagrange. Manifestement, l'action (2.12) ne se trouve pas sous une forme appropriée – les contraintes n'apparaissent pas explicitement. En outre, dans le cadre du formalisme hamiltonien, le temps, séparé des autres variables, est considéré comme un paramètre. La première chose à faire est donc de réécrire l'action (2.12) en scindant l'espace et le temps, c'est-à-dire en utilisant une décomposition $3+1$ de l'espace-temps. Ce faisant, nous verrons que l'action peut s'écrire sous la forme hamiltonienne ADM :

$$S = \int \left[-g_{ij}\frac{\partial \pi^{ij}}{\partial t} - NC_0 - N^iC_i - 2\Big(\pi^{ij}N_j - \frac{1}{2}N^i\mathrm{Tr}(\pi) + N^{|i}\sqrt{g}\Big)_{,i}\right] d^4x \tag{2.13}$$

qui nous permettra de déduire les contraintes hamiltoniennes.

2.3.1 Ecriture de l'action des théories tenseurs-scalaires à l'aide de la décomposition $3+1$ de l'espace-temps

Décomposition $3+1$ de l'espace temps

La décomposition $3+1$ de l'espace-temps consiste à le séparer en une série d'hypersurfaces spatiales paramétrisées par la variable temporelle t. Commençons par définir les fonctions *lapse* et *shift*.

Soit deux hypersurfaces $\Sigma(t)$ et $\Sigma(t+dt)$ représentées sur la figure 2.1 et leurs 3-métriques, respectivement ${}^{(3)}g_{ij}(t,x^k)\,dx^i\,dx^j$ et ${}^{(3)}g_{ij}(t+dt,x^k)\,dx^i\,dx^j$. Soit le point P_1 de coordonnées (x^i,t) sur $\Sigma(t)$. Nous définissons le point P_2 comme étant l'intersection de $\Sigma(t+dt)$ et de la normale à $\Sigma(t)$ en P_1. L'intervalle de temps propre $d\tau = N\,dt$ entre P_1 et P_2 définit alors la *fonction lapse* $N(x_k,t)$. Définissons le point P_3 de $\Sigma(t+dt)$ comme étant le point de cette hypersurface possédant les mêmes coordonnées spatiales que le point P_1. Le point P_3 a donc pour coordonnées $(x^i,t+dt)$ et le point P_2, les coordonnées $(x^i - N^i\,dt,t+dt)$. Le vecteur qui relie P_2 et P_3 définit alors les *fonctions shift* $N^i(x_k,t)$. Soit le point P_4 de $\Sigma(t+dt)$ de coordonnées $(x^i+dx^i,t+dt)$ et le point P_6 de $\Sigma(t)$ possédant les mêmes coordonnées spatiales que le point P_4, soit (x^i+dx^i,t). On définit P_5 comme étant l'intersection de la normale de $\Sigma(t+dt)$ en P_4 avec $\Sigma(t)$. Les coordonnées de P_5 sont alors $(x^i+dx^i+N^i\,dt,t)$.

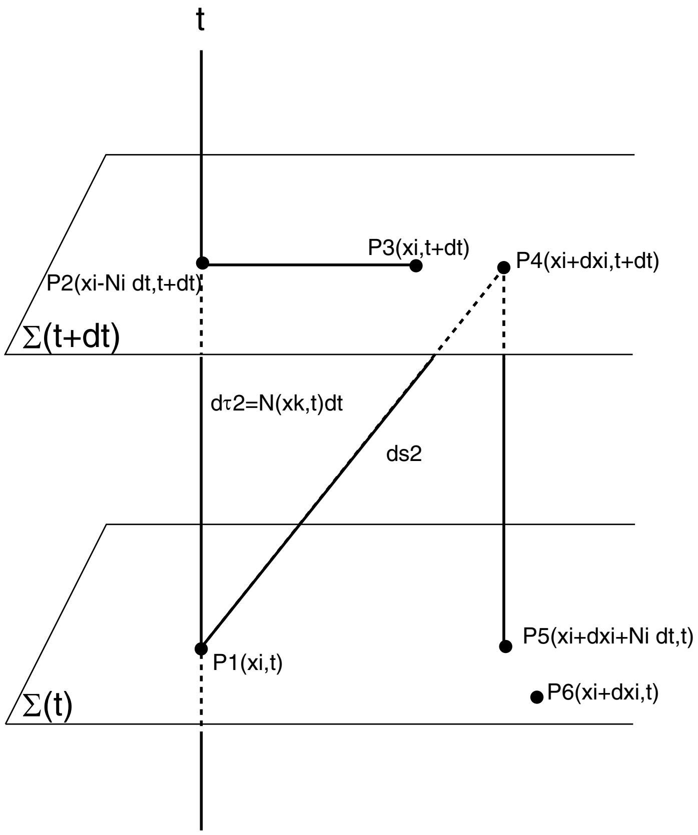


FIG. 2.1 – *La décomposition $3+1$ de l'espace temps.*

On peut désormais exprimer l'intervalle de longueur ds^2 entre les points P_1 et P_4 à l'aide de la 3-métrique ${}^{(3)}g_{ij}$ en termes des fonctions *shift* et *lapse*. Ecrivant le théorème de Pythagore dans le 4-espace non euclidien de signature $(-,+,+,+)$, il vient :

$$\begin{aligned}
ds^2 &= {}^{(4)}g_{\alpha\beta}\,dx^\alpha\,dx^\beta \\
&= {}^{(3)}g_{ij}(x^k,t)\big(x^i(P_5)-x^i(P_1)\big)\big(x^j(P_5)-x^j(P_1)\big) - d\tau^2 \\
&= {}^{(3)}g_{ij}(x^k,t)(dx^i+N^i\,dt)(dx^j+N^j\,dt) - N^2\,dt^2
\end{aligned}$$

ce qui nous donne pour la métrique :

$${}^{(4)}g_{\alpha\beta} = \begin{pmatrix} {}^{(4)}g_{00} & {}^{(4)}g_{0j} \\ {}^{(4)}g_{0i} & {}^{(4)}g_{ij} \end{pmatrix} = \begin{pmatrix} -N^2 + {}^{(3)}g_{ij}N^iN^j & {}^{(3)}g_{ij}N^i \\ {}^{(3)}g_{ij}N^j & {}^{(3)}g_{ij} \end{pmatrix}$$

soit, en posant $N_i \doteq {}^{(3)}g_{ij}N^j$:

$$ {}^{(4)}g_{\alpha\beta} = \begin{pmatrix} N_k N^k - N^2 & N_j \\ N_i & {}^{(3)}g_{ij} \end{pmatrix} \tag{2.14} $$

et en servant du fait que ${}^{(4)}g_{\alpha\beta}{}^{(4)}g^{\beta\gamma} = \delta^\gamma_\alpha$:

$$ {}^{(4)}g^{\alpha\beta} = \begin{pmatrix} -N^{-2} & N_j N^{-2} \\ N_i N^{-2} & {}^{(3)}g_{ij} - N^i N^j N^{-2} \end{pmatrix} \tag{2.15} $$

Pour calculer le déterminant ${}^{(4)}g$ de la 4-métrique, on peut se servir du théorème de Frobenius-Schur qui montre que si A, B, C et D sont quatre matrices carrées, le déterminant de la matrice :

$$ \Delta \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.16} $$

est $\det(\Delta) = \det(D)\det(A - BD^{-1}C)$. On en déduit donc que :

$$ \sqrt{-{}^{(4)}g} = N\sqrt{{}^{(3)}g} \tag{2.17} $$

Avant de poursuivre, il nous faut définir le concept de *courbure extrinsèque* qui caractérise la courbure d'une hypersurface immergée dans une variété de dimension supérieure (par exemple d'une hypersurface spatiale que l'on plonge dans une 4-géométrie). La courbure extrinsèque caractérise la manière dont une variété est incluse dans un espace de dimension supérieure. Par exemple, une feuille de papier à deux dimensions que l'on tord dans un espace à trois dimensions possède une courbure extrinsèque relativement à cet espace. Au contraire, notre Univers possède une *courbure intrinsèque* qui ne nécessite pas de dimensions supplémentaires pour être définie. Dans le cas qui nous intéresse ici, la courbure extrinsèque d'une hypersurface spatiale est une mesure de la variation de direction de la normale $\vec{n}$ à l'hypersurface $\Sigma(t)$ entre deux points infiniment voisins sur $\Sigma(t)$ et est définie par :

$$ K_{ij} \doteq -n_{i;j} = -\vec{e}_j \cdot \nabla_i \vec{n} \tag{2.18} $$

Les relations de Gauss-Weingarten et de Gauss-Codazzi : réécriture de l'action

Dès que la courbure extrinsèque est connue, on peut exprimer la dérivée covariante des vecteurs de base $\vec{e}_j$ de l'hypersurface spatiale Σ dans l'espace-temps, en termes de quantités qui dépendent de Σ seule. Ce sont les relations de Gauss-Weingarten qui s'écrivent :

$$ {}^{(4)}\nabla_i \vec{e}_j = -K_{ij}\vec{n} + {}^{(3)}\Gamma^k_{ij}\vec{e}_k \tag{2.19} $$

Les relations de Gauss-Codazzi, quant à elles, tentent d'exprimer la courbure intrinsèque de l'espace-temps en fonction des courbures intrinsèque et extrinsèque de l'hypersurface. Elles s'expriment comme :

$$ {}^{(4)}R^0_{ijk} = K_{ik|j} - K_{ij|k} \tag{2.20} $$

$$ {}^{(4)}R_{mijk} = -(K_{ij}K_{mk} - K_{ik}K_{mj}) + {}^{(3)}R_{mijk} \tag{2.21} $$

$$ {}^{(4)}R_{i0k0} = K_{ik,n} + K^m_k K_{im} \tag{2.22} $$

où $|$ désigne la dérivée covariante dans l'hypersurface. Ces relations nous permettent de réécrire la courbure scalaire en fonction des courbures intrinsèque et extrinsèque de l'hypersurface. En effet, on peut écrire :

$$ {}^{(4)}R = {}^{(4)}R^{ij}_{ij} - 2\,{}^{(4)}R^j_{0j0} $$

Le membre de droite ne contenant que des termes explicités par les relations de Gauss-Codazzi et définissant $\mathrm{Tr}(K^2) \doteq K^{jk}K_{jk}$ et $K \doteq K^i_i \doteq \mathrm{Tr}(K)$, on calcule que :

$$ {}^{(4)}R^{ij}_{ij} = g^{ik}g^{jm}\,{}^{(4)}R_{kmij} = {}^{(3)}R - \mathrm{Tr}(K^2) + K^2 $$

et :

$$ {}^{(4)}R^j_{0j0} = g^{jk}\,{}^{(4)}R_{k0j0} = K_{,n} + \mathrm{Tr}(K^2) - K_{kj}g^{jk}_{,n} $$

Comme, de plus, on peut montrer que :

$$ g^{jk}_{,n} = 2K^{jk} $$

il vient alors pour la courbure scalaire :

$$ {}^{(4)}R = {}^{(3)}R + \mathrm{Tr}(K^2) + K^2 - 2K_{,n} \tag{2.23} $$

Se servant de cette dernière relation et de (2.17), nous pouvons réécrire l'action (2.12) de Hilbert de la manière suivante :

$$ S = \int_M N\sqrt{{}^{(3)}g}\big({}^{(3)}R + \mathrm{Tr}(K^2) - K^2\big)\, d^4x - 2N\int_{\partial M} K\sqrt{{}^{(3)}g}\, d^3x \tag{2.24} $$

Le terme de surface peut être éliminé en imposant des conditions spécifiques à la frontière de la variété ou en ajoutant à l'action de départ un autre terme de surface compensant celui de (2.24) – dans ce dernier cas, on élimine le terme de surface en prenant comme action de départ :

$$ S = \int_M {}^{(4)}R\sqrt{-{}^{(4)}g}\, d^4x + 2\int_{\partial M} K\sqrt{{}^{(3)}g}\, d^3x \tag{2.25} $$

Dans le cas d'espaces fermés, l'élimination du terme de surface ne pose aucun problème (il est sans influence sur les équations de champs lorsque l'on varie la géométrie à l'intérieur de la surface frontière de la variété). Pour des espaces ouverts asymptotiquement plats, par contre, il est nécessaire d'ajouter un terme de surface. Quoi qu'il en soit, de la façon dont on se débarrasse de ce terme de surface, on obtient :

$$ S = \int_M N\sqrt{{}^{(3)}g}\big({}^{(3)}R + \mathrm{Tr}(K^2) - K^2\big)\, d^4x \tag{2.26} $$

Dans ce qui suit, nous allons réécrire l'action (2.25) à l'aide de la décomposition $3+1$.

Ecriture de l'action sous la forme $3+1$

Afin d'écrire l'action ci-dessus sous la forme d'une décomposition $3+1$, il nous faut réaliser cette transformation pour :

1. les symboles de Christoffel qui s'écrivent :

$$ \Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2}g^{\gamma\delta}\big(g_{\alpha\delta,\beta} + g_{\delta\beta,\alpha} - g_{\alpha\beta,\delta}\big) $$

2. le tenseur de Ricci qui s'écrit :

$$ {}^{(4)}R_{ij} = \Gamma^{\alpha}_{ij,\alpha} - \Gamma^{\alpha}_{i\alpha,j} + \Gamma^{\alpha}_{ij}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\alpha}_{i\beta}\Gamma^{\beta}_{j\alpha} $$

3. la courbure scalaire qui s'écrit ${}^{(4)}R = {}^{(4)}R^{\alpha}_{\alpha}$, et donc l'action.

Afin d'écrire les symboles de Christoffel Γ, nous introduisons la définition suivante :

$$ \xi^j \doteq \frac{N^j}{N} $$

et définissons les composantes du tenseur Λ, les 3-symboles de Christoffel relatifs à l'hypersurface, comme :

$$ \Lambda^j_{ik} \doteq \frac{1}{2}{}^{(3)}g^{jm}\big({}^{(3)}g_{im,k} + {}^{(3)}g_{mk,i} - {}^{(3)}g_{ik,m}\big) $$

Après de longs calculs, on obtient les formes suivantes des symboles de Christoffel :

$$ \begin{aligned} \Gamma^0_{00} &= \frac{1}{N}\partial_0 N + N_{|i}\xi^i - N\xi^i\xi^j K_{ij} \\ \Gamma^i_{00} &= N\gamma^{ij}\partial_0\xi_j + \frac{1}{2}\gamma^{ij}\left[N^2(1-\xi_m\xi^m)\right]_{,j} - NN_{|j}\xi^i\xi^j + N^2\xi^i\xi^j\xi^k K_{jk} \\ \Gamma^j_{ik} &= \Lambda^j_{ik} + \xi^j K_{ik} \\ \Gamma^0_{0i} &= \frac{N_{|i}}{N} - K_{ij}\xi^j \\ \Gamma^j_{i0} &= N(-K^j_i + \xi^j_{|i} + \xi^j K_{im}\xi^m) \end{aligned} $$

On les injecte alors dans l'expression des composantes spatiales du tenseur de Ricci ${}^{(4)}R_{ij}$ qui s'expriment en fonction des symboles de Christoffel comme :

$$ {}^{(4)}R_{ij} = \Gamma^{\alpha}_{ij,\alpha} - \Gamma^{\alpha}_{i\alpha,j} + \Gamma^{\alpha}_{ij}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\alpha}_{i\beta}\Gamma^{\beta}_{j\alpha} $$

Il vient donc :

$$ \begin{aligned} {}^{(4)}R_{ij} &= {}^{(3)}R_{ij} - \frac{1}{N}\partial_0 K_{ij} - \frac{1}{N}\left(N_{|ij} - N_{|i}K_{jk}\xi^k - N_{|j}K_{ik}\xi^k\right) \\ &\quad + K_{ij}K - 2K_{ik}K^k_j + K_{ik}\xi^k_{|j} + K_{jk}\xi^k_{|i} + \xi^k K_{ij|k} \end{aligned} \tag{2.27} $$

où ${}^{(3)}R_{ij}$ est le tenseur de Ricci d'une hypersurface et est donc défini de manière conventionnelle en fonction de ses symboles de Christoffel :

$$ {}^{(3)}R_{ij} = \Lambda^k_{ij,k} - \Lambda^k_{ik,j} + \Lambda^k_{ij}\Lambda^l_{kl} - \Lambda^k_{il}\Lambda^l_{jk} $$

Pour poursuivre notre calcul, nous devrions également écrire explicitement les composantes ${}^{(4)}R_{0i}$ et ${}^{(4)}R_{00}$ du tenseur de Ricci. Cependant, ces calculs sont nettement plus laborieux que ceux qui conduisent aux composantes purement spatiales (2.27) du tenseur de Ricci. En fait, il suffit d'utiliser un système de référence qui simplifie le calcul sans pour autant occulter les informations relatives à la liberté de choix du système de référence, en relativité générale. Nous utiliserons le système de référence défini par les relations :

$$ \begin{aligned} \vec{n} &\doteq \frac{1}{N}\frac{\partial}{\partial t} - \frac{N^i}{N}\frac{\partial}{\partial x^i} \\ \vec{e}_i &\doteq \frac{\partial}{\partial x^i} \end{aligned} $$

Dans ce système particulier, on a $g_{nn} = \vec{n}\cdot\vec{n} = -1$, $g_{ni} = \vec{n}\cdot\vec{e}_i = 0$, $g_{ij} = \vec{e}_i\cdot\vec{e}_j$. La courbure scalaire ${}^{(4)}R$ s'écrit alors ${}^{(4)}R = 2G^n_n + 2g^{ij\,(4)}R_{ij}$, où $G_{\alpha\beta}$ désigne le tenseur d'Einstein. En vertu des relations de Gauss-Codazzi (2.21), on obtient :

$$ G^n_n = -\frac{1}{2}\left({}^{(3)}R + K^2 - K_{ij}K^{ij}\right) $$

On introduit cette expression et (2.27) dans la courbure scalaire donnée par ${}^{(4)}R = 2G^n_n + 2g^{ij\,(4)}R_{ij}$, afin d'obtenir :

$$ \begin{aligned} {}^{(4)}R &= {}^{(3)}R + K^2 - K_{ij}K^{ij} - \frac{2}{N}g^{ij}\partial_0 K_{ij} - \frac{2}{N}N^{|i}_{|i} + \frac{4}{N}N_{|i}K^i_k\xi^k \\ &\quad -2K_{ij}K^{ij} + 4K^i_k\xi^k_{|i} + 2g^{ij}K_{ij|k}\xi^k \end{aligned} $$

où ${}^{(3)}R$ est le scalaire de courbure relatif à l'hypersurface ; il s'exprime comme :

$$ {}^{(3)}R = 2\,{}^{(3)}R^{12}_{12} + 2\,{}^{(3)}R^{13}_{13} + 2\,{}^{(3)}R^{23}_{23} $$

Finalement, l'action dans la décomposition $3+1$ prendra la forme suivante :

$$ \begin{aligned} S &= \int_M d^4x\,\sqrt{g}\Big[-g^{ij}\frac{\partial K_{ij}}{\partial t} - \frac{\partial K}{\partial t} + N\left({}^{(3)}R + K^2 - K_{ij}K^{ij}\right) + 2N^i\delta^j_i K_{|j} \\ &\quad -2N^{|i}_{|i} + 2N^{|i}K_{ij}\xi^j + 2NK_{ij}\xi^{j|i}\Big] \end{aligned} \tag{2.28} $$

2.3.2 Identification de la forme $3+1$ de l'action avec sa forme dans le formalisme hamiltonien

Nous désirons maintenant démontrer l'équivalence entre la forme (2.28) de l'action et sa forme adaptée au formalisme hamiltonien :

$$ S = \int_M d^4x\left[-g_{ij}\frac{\partial \pi^{ij}}{\partial t} - NC_0 - N^iC_i - 2\left(\pi^{ij}N_j - \frac{1}{2}N^i\mathrm{Tr}(\pi) + N^{|i}\sqrt{g}\right)_{,i}\right] \tag{2.29} $$

Pour cela, nous définissons les moments canoniquement conjugués π^{ij} :

$$ \pi^{ij} \doteq \sqrt{g}(g^{ij}K - K^{ij}) \tag{2.30} $$

et introduisons le superhamiltonien comme :

$$C_0 = -({}^{(3)}R + K^2 - K_{ij}K^{ij})\sqrt{g} = -\sqrt{g}{}^{(3)}R + \sqrt{g}(K_{ij}K^{ij} - K^2) \tag{2.31}$$

Or, on calcule que :

$$K_{ij}K^{ij} - K^2 = \frac{1}{g}\left[\mathrm{Tr}(\pi^2) - \frac{1}{2}\big(\mathrm{Tr}(\pi)\big)^2\right]$$

et par conséquent, il vient :

$$C_0 = -\sqrt{g}{}^{(3)}R + \frac{1}{\sqrt{g}}\left[\mathrm{Tr}(\pi^2) - \frac{1}{2}\big(\mathrm{Tr}(\pi)\big)^2\right] \tag{2.32}$$

D'autre part, on peut également montrer que :

$$- g_{ij}\frac{\partial \pi^{ij}}{\partial t} = \sqrt{g}\left(-g^{ij}\frac{\partial K_{ij}}{\partial t} - \frac{\partial K}{\partial t}\right) \tag{2.33}$$

Enfin, on introduit les supermoments C_i *via* :

$$- N_iC^i = -2N^i(K_i^j - \delta_i^j K)_{|j}\sqrt{g} = 2N^i\big[-\sqrt{g}(K_i^j - \delta_i^j K)\big]_{|j} \tag{2.34}$$

que l'on peut réécrire :

$$\begin{aligned} C_i &= -2\pi^j_{i|j} = 2\sqrt{g}(K_i^j - \delta_i^j K)_{|j} \\ C^i &= -2\pi^{ij}_{|j} = 2\sqrt{g}(K^{ij} - g^{ij}K)_{|j} \end{aligned}$$

En insérant l'ensemble de ces résultats dans l'action (2.29), on retrouve bien, après quelques calculs supplémentaires, l'action écrite à l'aide de la décomposition $3 + 1$.

2.3.3 Formulation des contraintes ADM de la relativité générale

Les moments canoniques π^{ij} sont naturellement définis par :

$$\pi^{ij} \doteq \frac{\delta L}{\delta \dot{g}_{ij}}$$

L étant le lagrangien. L'action du formalisme hamiltonien s'écrit :

$$S = \int_M \left(\pi^{ij}\frac{\partial g_{ij}}{\partial t} - NC_0 - N^iC_i\right) d^4x \tag{2.35}$$

La démonstration de l'équivalence entre l'expression ci-dessus et la forme (2.12) de l'action revient à assurer l'équivalence entre les actions (2.26) et (2.13). Par conséquent, en variant (2.35) par rapport aux fonctions *lapse* et *shift*, les multiplicateurs de Lagrange, on obtient les contraintes :

$$C_0 = -\sqrt{g}{}^{(3)}R + \frac{1}{\sqrt{g}}\left[\mathrm{Tr}(\pi^2) - \frac{1}{2}\big(\mathrm{Tr}(\pi)\big)^2\right] = 0 \tag{2.36}$$

$$C_i = -2\pi^j_{i|j} = 0 \tag{2.37}$$

Les contraintes gouvernent la dynamique de la géométrie et constituent, dans le même temps, des conditions aux valeurs initiales. En raison des contraintes $C_\mu = 0$, il est impossible de choisir librement les champs ${}^{(3)}g_{ij}$ et π^{ij} sur l'hypersurface initiale $\Sigma(t_0)$. Les équations dynamiques dictent le changement de la géométrie intrinsèque et de la courbure extrinsèque d'une hypersurface lorsque l'on se déplace d'une hypersurface à une hypersurface voisine. Si les contraintes sont satisfaites sur $\Sigma(t_0)$ et si les champs canoniquement conjugués évoluent en vérifiant les équations dynamiques, alors les contraintes sont conservées dans le temps.

En vertu des contraintes, quatre des champs ${}^{(3)}g_{ij}$ et π^{ij} peuvent être exprimés en fonction des autres ; en outre, l'imposition des conditions de coordonnées fixe quatre des champs restants. Seules demeurent deux paires de variables canoniques ; le paragraphe suivant montre comment l'approche ADM se ramène à ces deux degrés de liberté physiques.

2.3.4 Formulation ADM des théories tenseurs-scalaires minimalement couplées et massives en l'absence de fluide parfait

Afin de savoir comment sont modifiées les équations de contraintes (2.36) et (2.37) en présence d'un champ scalaire, nous allons écrire la décomposition $3+1$ de la partie de l'action comprenant le champ scalaire :

$$S = \int \left[R - \frac{1}{2}\frac{2\omega+3}{\phi^2}\phi_{,\mu}\phi^{,\mu} - U + 16\pi L_m \right] \sqrt{-^{(4)}g}\, d^4x$$

Dans un premier temps, nous réécrivons le terme contenant la fonction de couplage ω de Brans-Dicke de la manière suivante :

$$-(3/2+\omega)\phi^{,\mu}\phi_{,\mu}\phi^{-2}\sqrt{-^{(4)}g} = (3/2+\omega)\dot{\phi}^2\phi^{-2}N^{-1}\sqrt{g} \tag{2.38}$$

En tenant compte de cette dernière expression, nous pouvons exprimer le moment conjugué du champ scalaire :

$$\pi_\phi \doteq \frac{\partial I}{\partial \dot{\phi}} = (3+2\omega)\dot{\phi}\phi^{-2}N^{-1}\sqrt{g} \tag{2.39}$$

où I est le Lagrangien de l'action ci-dessus. De cette dernière équation, on déduit :

$$\dot{\phi} = \pi_\phi \frac{N}{\sqrt{g}}\frac{\phi^2}{3+2\omega} \tag{2.40}$$

ce qui nous permet de réécrire (2.38) de la manière suivante :

$$\begin{aligned} -(3/2+\omega)\phi^{,\mu}\phi_{,\mu}\phi^2\sqrt{^{(4)}g} &= \frac{(3/2+\omega)}{\phi^2}\frac{1}{N}\sqrt{g}\pi_\phi^2\frac{N^2}{g}\frac{\phi^4}{(3+2\omega)^2} \\ &= \frac{1}{2}\frac{\phi^2}{3+2\omega}\frac{N}{\sqrt{g}}\pi_\phi^2 \end{aligned}$$

A la contrainte (2.36) vont donc venir s'ajouter des termes $C_{0\phi}$ issus de la présence d'une fonction de couplage et d'un potentiel, $C_{0\phi}$ étant tel que :

$$\frac{1}{2}\frac{\phi^2}{3+2\omega}\frac{N}{\sqrt{g}}\pi_\phi^2 - NU\sqrt{g} = \pi_\phi\dot{\phi} - NC_{0\phi} \tag{2.41}$$

D'où, en se servant de l'expression de π_ϕ donnée par (2.39) et de $\dot{\phi}$, il vient :

$$\begin{aligned} C_{0\phi} &= -\frac{1}{2}\frac{\phi^2}{3+2\omega}\frac{1}{\sqrt{g}}\pi_\phi^2 + \pi_\phi\frac{\dot{\phi}}{N} + U\sqrt{g} \\ &= -\frac{1}{2}\frac{\phi^2}{3+2\omega}\frac{1}{\sqrt{g}}\pi_\phi^2 + \frac{\phi^2}{3+2\omega}\frac{1}{\sqrt{g}}\pi_\phi^2 + U\sqrt{g} \\ &= \frac{1}{2}\frac{\phi^2}{3+2\omega}\frac{1}{\sqrt{g}}\pi_\phi^2 + U\sqrt{g} \end{aligned}$$

La forme finale de la contrainte C_0, en tenant compte de la présence du champ scalaire, est donc :

$$C_0 = -\sqrt{g}^{(3)}R + \frac{1}{\sqrt{g}}\left[\mathrm{Tr}(\pi^2) - \frac{1}{2}\big(\mathrm{Tr}(\pi)\big)^2\right] + \frac{1}{2}\frac{\phi^2}{3+2\omega}\frac{1}{\sqrt{g}}\pi_\phi^2 + U\sqrt{g}$$

Si l'on utilise la métrique suivante :

$$ds^2 = -N^2 d\Omega + R_0^2 e^{-2\Omega}\left(e^{\beta_+ + \sqrt{3}\beta_-}(\omega^1)^2 + e^{\beta_+ - \sqrt{3}\beta_-}(\omega^2)^2 + e^{-2\beta_+}(\omega^3)^2\right)$$

et la paramétrisation de Misner [24, **?**] :

$$\begin{aligned} p^i_k &= 2\pi\pi^i_k - \frac{2}{3}\pi\delta^i_k\pi^l_l \\ 6p_{ij} &= \mathrm{diag}\big(p_+ + \sqrt{3}p_-, p_+ - \sqrt{3}p_-, -2p_+\big) \end{aligned}$$

cette contrainte s'écrit :

$$\begin{aligned} C_0 &= -R_0^3 e^{-3\Omega}\left[{}^{(3)}R + \frac{1}{R_0^6 e^{-6\Omega}}\left(\frac{1}{6}(\pi_k^k)^2 - \frac{1}{24\pi^2}(p_+^2 + p_-^2)\right)\right] \\ &\quad + \frac{1}{2R_0^3 e^{-3\Omega}}\frac{\phi^2\pi_\phi^2}{(3+2\omega)} + R_0^3 e^{-3\Omega}U \end{aligned}$$

π étant le nombre dans cette dernière expression, et l'action devient :

$$S = \int p_+\, d\beta_+ + p_-\, d\beta_- + p_\phi\, d\phi - H\, d\Omega$$

avec l'hamiltonien ADM $H = 2\pi\pi_k^k$ et $p_\phi = \pi\Pi_\phi$. En utilisant la contrainte $C_0 = 0$, on peut alors déduire pour H :

$$H^2 = p_+^2 + p_-^2 + 12\frac{p_\phi^2\phi^2}{3+2\omega} + 24\pi^2 R_0^6 e^{-6\Omega}U \tag{2.42}$$

Les degrés de liberté physiques ont été isolés, mais sous cette forme la théorie n'est plus covariante – les contraintes ont été résolues et les conditions de coordonnées fixées. La perte de covariance est d'ailleurs patente : le hamiltonien ADM n'est pas nul, tandis que l'annulation du hamiltonien est caractéristique des systèmes contraints.

Troisième partie

Différentes méthodes pour l’étude des cosmologies homogènes en théories tenseur-scalaires

Chapitre 1

Introduction

Cette partie constitue la première étape de notre travail de thèse: nous cherchons une méthode unifiée nous permettant d'étudier un maximum de modèles homogènes afin de contraindre un large éventail de théories tenseur-scalaires.
De telles méthodes existent qui permettent d'étudier tous les modèles de Bianchi mais pas toutes les théories tenseur-scalaires. Ainsi, le formalisme Hamiltonien ADM à permis d'analyser l'approche de la singularité des modèles de Bianchi de la classe A en Relativité Générale dans le vide à l'aide d'une représentation quantitative d'un point heurtant des murs de potentiels. Les méthodes d'analyse dynamique d'Ellis et Wainwright[25] permettent de décrire l'évolution d'un Univers à l'aide de ses points d'équilibre et s'appliquent à des modèles homogènes comme inhomogènes, avec un fluide parfait ou tilté, etc. Cependant, nous n'avons pas trouvé dans la littérature de méthodes permettant d'obtenir des résultats généraux sur l'évolution des cosmologies homogènes quelque soit la théorie tenseur-scalaire envisagée. Par exemple, dans la recherche de conditions menant un Univers anisotrope vers l'isotropie, les méthodes employées sont le plus souvent spécifiques à une théorie tenseur-scalaire précise mais inadaptables pour d'autres.
Dans ce qui suit, nous allons présenter à travers une série d'articles, les différentes méthodes dont nous nous sommes servis pour tenter de contraindre les théories tenseur-scalaires à travers l'étude des cosmologies homogènes. Dans les chapitres 2 et 3 nous rechercherons les solutions exactes des équations de champs en se donnant respectivement une théorie tenseur-scalaire et en calculant ses solutions ou vice-versa. Dans le chapitre 4, on s'intéresse à l'évolution dynamique de l'Univers par rapport à la dépendance des signes des dérivées premières et secondes des fonctions métriques vis à vis d'une théorie tenseur-scalaire définie par une fonction de Brans-Dicke ω et une fonction de gravitation ϕ^{-1}. Ces résultats sont généralisés dans le chapitre 5 pour une fonction de gravitation inconnue. Dans le chapitre 6, on étudie une théorie tenseur scalaire massive à l'aide du formalisme Hamiltonien et on recherche les conditions pour que l'Univers soit asymptotiquement en expansion et isotrope. Dans le chapitre 7, on considère l'existence d'un singularité initiale pour une théorie tenseur-scalaire non minimalement couplée et sans masse, en examinant la divergence des scalaires de courbure, de Ricci et de Kretchmann. Le chapitre 8 examine les contraintes portant sur les théories tenseur-scalaires lorsqu'on leur impose une symétrie de Noether.

Chapitre 2

Solutions exactes pour le modèle de Bianchi de type I: en se donnant des fonctions du champ scalaire(1 article)

La méthode la plus directe pour étudier les modèles cosmologiques homogènes en présence de champs scalaires est indéniablement la recherche des solutions exactes d'une théorie tenseur-scalaire complètement définie, c'est-à-dire dont toutes les fonctions du champ scalaire sont connues. Certaines solutions exactes jouent un rôle considérable en cosmologie. Ainsi la solution de Kasner concernant le modèle de Bianchi de type I en Relativité Générale sert à décrire le comportement asymptotique de nombreux modèles homogènes à l'approche de la singularité. La solution de Schwarshild caractérise les effondrements gravitationnels. La solution de De Sitter est incontournable dans de nombreux modèles inflationnaires. Le modèle de Lemaître présente une période d'expansion décélérée suivie d'une période accélérée, transition récemment détectée pour notre Univers. Il en existe évidement bien d'autres.
Dans l'article qui suit nous allons aborder la recherche de solutions exactes pour le modèle de Bianchi de type I et pour une théorie tenseur-scalaire non minimalement couplée définie par:

$$L = -\phi R + \omega \phi_{,\alpha} \phi^{,\alpha} \phi^{-1}$$

où ω est la fonction de Brans-Dicke dont nous considèrerons les formes particulières suivantes:

$$\begin{aligned} 3 + 2\omega(\phi) &= 2\beta(1 - \phi/\phi_c)^{-\alpha} \\ 3 + 2\omega(\phi) &= \phi_c^2 \phi^{2m} \\ 3 + 2\omega(\phi) &= e^{2\phi_c \phi} \end{aligned}$$

Afin de simplifier la recherche de solutions exactes, il est possible et même souvent recommandé d'utiliser une transformation conforme de la métrique $g_{\alpha\beta}$ définie par:

$$\tilde{g}_{\alpha\beta} = \phi g_{\alpha\beta}$$

Elle a pour effet de rendre le champ scalaire minimalement couplé: le Lagrangien devient celui de la Relativité Générale plus un champ scalaire minimalement couplé, à savoir:

$$L = R - \frac{3 + 2\omega}{\phi^2} \phi_{,\alpha} \phi^{,\alpha}$$

Le référentiel décrit par les fonctions métriques $g_{\alpha\beta}$ est traditionnellement appelé référentiel de Brans-Dicke alors que celui décrit par $\tilde{g}_{\alpha\beta}$ est appelé référentiel d'Einstein.
Trouver des solutions exactes physiquement intéressantes n'est pas une tache facile d'autant plus que la tendance est à la complexification des théories et de leur contenu: variation de la constante de gravitation, du potentiel, de la vitesse de la lumière, prise en compte de fluide plus complexes que les fluides parfaits(dissipatif, gaz de chaplygin), radiation noire provenant des cosmologies branaires. Bien souvent, il apparaît même difficile de préciser ce que serait une solution physiquement intéressante tellement les possibilités sont grandes et le danger est de ne produire qu'une solution dont l'intérêt est purement mathématique. Face à la diversité que présentent les théories tenseur-scalaires, la recherche de solutions exactes n'apparaît donc pas comme étant la méthode la mieux adaptée pour contraindre ces théories.

Generalised scalar-tensor theory in the Bianchi type I model

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Abstract

We use a conformal transformation to find solutions to the generalised scalar-tensor theory, with a coupling constant dependent on a scalar field, in an empty Bianchi type I model. We describe the dynamical behaviour of the metric functions for three different couplings: two exact solutions to the field equations and a qualitative one are found. They exhibit non-singular behaviours and kinetic inflation. Two of them admit both General Relativity and string theory in the low-energy limit as asymptotic cases.

Key words: Bianchi models; Generalised scalar-tensor theory; Exact solution; non-singular Universe; Kinetic inflation.

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2.1 Introduction

Scalar-tensor theories seem to be essential to describe gravitational interactions near the Plank scale : string theory, higher order theories in the Ricci scalar [26], extended inflation and many others theories imply scalar field.

The generalised scalar-tensor Lagrangian has the same form as the Brans-Dicke theory [7] but with a coupling constant ω depending on the scalar field. Such a theory is interesting for many reasons. Hence, if we choose ω as a constant, the Lagrangian is identical to Brans-Dicke Lagrangian. This theory tends to General Relativity for large value of the coupling constant ($\omega > 500$). But, if we choose $\omega = -1$, the Brans-Dicke theory is identical to the string theory in the low-energy limit. Hence, the generalised scalar-tensor theory seems to be able to build a " bridge " between string theory and General Relativity. Other reasons, as inflation, can be put forward : such a theory with a varying coupling constant, may drive the scale factors to accelerate without potential or cosmological constant [27, 28], i.e. called kinetic inflation.

The generalised scalar-tensor theory has been studied by many authors and the method we will use to find exact solutions has always been described in [29] in the presence of matter in the Lagrangian. Here, we will consider the empty Bianchi type I Universe, which is spatially flat, and will use three different forms of the coupling constant $\omega(\phi)$. The first form, $2\omega(\phi) + 3 = 2\beta(1 - \phi/\phi_c)^{-\alpha}$, has been introduced by Garcia-Bellido and Quiros [30] and studied by Barrow [31] in the context of a FLRW flat model with vacuum or radiation. It has also been studied in [29], for a Bianchi type I model, where a solution is found in presence of matter. In this paper, we will write explicitly an exact solution and will study the dynamical behaviour of the metric functions which depends on the integration constant. We will cast light on interesting features such as kinetic inflation. The second form is a power law type, $3+2\omega(\phi) = \phi_c^2\phi^{2m}$. Here again, we will give explicitly an exact solution and study it. An interesting feature is the possibility of a non-singular Universe. The third form is an exponential law type, $3 + 2\omega(\phi) = e^{2\phi_c\phi}$ and will be studied qualitatively. These two last laws seem interesting because power and exponential laws are very present in physics. They play a fundamental role for the metric functions of course, but also when we consider a potential $V(\phi)$ [32] giving birth to extended or chaotic inflation [33]. Moreover, we will see how the power law form of the coupling constant is linked to minimally coupled and induced gravity for large or small values of the scalar field.

This paper is organised as follows. In section 2.2, we write field equations in both Brans-Dicke and conformal frame and explain how to proceed to solve them. In section 2.3, we derive solution for each of the three forms of $\omega(\phi)$ and study them.

2.2 Field equations.

2.2.1 Field equations in the Brans-Dicke frame.

We work with the metric:

$$ds^2 = -dt^2 + a(t)^2(\omega^1)^2 + b(t)^2(\omega^2)^2 + c(t)^2(\omega^3)^2 \tag{2.1}$$

$a(t)$, $b(t)$, $c(t)$ are the metric functions, ω^i are the 1-forms of the Bianchi type I model and t, the proper time. We express the Lagrangian of the theory in the form:

$$L = -\phi R + \omega(\phi)\phi_{,\alpha}\phi^{,\alpha}\phi^{-1} \tag{2.2}$$

One can also cast (2.2) on the form:

$$L = -f(\Phi)R + 1/2\partial_\alpha\Phi\partial^\alpha\Phi \tag{2.3}$$

with

$$\omega(\phi) = 1/2ff^{,-2} \tag{2.4}$$

The corresponding field equations and Klein-Gordon equation are obtained by varying the action (2.2) with respect to the space-time metric and the scalar field. If we introduce the τ time through

$$abcd\tau = dt \tag{2.5}$$

then, denoting $d/d\tau$ by a prime, the field equations are:

$$\begin{aligned}
\frac{a''}{a} - \frac{a'^2}{a^2} + \frac{a'}{a}\frac{\phi'}{\phi} - \frac{1}{2}\frac{\omega'}{3+2\omega}\frac{\phi'}{\phi} &= 0\\
\frac{b''}{b} - \frac{b'^2}{b^2} + \frac{b'}{b}\frac{\phi'}{\phi} - \frac{1}{2}\frac{\omega'}{3+2\omega}\frac{\phi'}{\phi} &= 0\\
\frac{c''}{c} - \frac{c'^2}{c^2} + \frac{c'}{c}\frac{\phi'}{\phi} - \frac{1}{2}\frac{\omega'}{3+2\omega}\frac{\phi'}{\phi} &= 0
\end{aligned} \tag{2.6}$$

$$\frac{a'}{a}\frac{b'}{b} + \frac{a'}{a}\frac{c'}{c} + \frac{b'}{b}\frac{c'}{c} + \frac{\phi'}{\phi}\left(\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c}\right) - \frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2 = 0 \tag{2.7}$$

$$\phi'' = -\frac{\omega'\phi'}{3+2\omega} \tag{2.8}$$

We can integrate (2.8) to obtain the useful equation:

$$A\phi'\sqrt{3+2\omega} = 1 \tag{2.9}$$

A being an integration constant. Hence, we see that $\omega > -3/2$.

2.2.2 Field equations in the conformal frame.

Now, we work with the conformal metric:

$$ds^2 = -d\tilde{t}^2 + \tilde{a}(t)^2(\omega^1)^2 + \tilde{b}(t)^2(\omega^2)^2 + \tilde{c}(t)^2(\omega^3)^2 \tag{2.10}$$

By the conformal transformation the metric has been redefined as:

$$\tilde{g}_{\alpha\beta} = \phi g_{\alpha\beta} \tag{2.11}$$

and the Lagrangian becomes:

$$L = R - 1/2(3+2\omega)\phi_{,\alpha}\phi^{,\alpha}\phi^{-2} \tag{2.12}$$

Hence, the generalised scalar-tensor theory is cast into Einstein gravity with a minimally coupled scalar field. In the $\tilde{\tau}$ time defined as:

$$\tilde{a}\tilde{b}\tilde{c}d\tilde{\tau} = d\tilde{t} \tag{2.13}$$

the field equations and the klein-Gordon equation become in the conformal frame

$$\begin{aligned}\frac{\tilde{a}''}{\tilde{a}'}-\frac{\tilde{a}'}{\tilde{a}} &= 0\\ \frac{\tilde{b}''}{\tilde{b}'}-\frac{\tilde{b}'}{\tilde{b}} &= 0\\ \frac{\tilde{c}''}{\tilde{c}'}-\frac{\tilde{c}'}{\tilde{c}} &= 0\end{aligned} \quad (2.14)$$

$$\frac{\tilde{a}'}{\tilde{a}}\frac{\tilde{b}'}{\tilde{b}}+\frac{\tilde{a}'}{\tilde{a}}\frac{\tilde{c}'}{\tilde{c}}+\frac{\tilde{b}'}{\tilde{b}}\frac{\tilde{c}'}{\tilde{c}} = \frac{1}{2}(\omega+3/2)(\frac{\phi'}{\phi})^2 \quad (2.15)$$

$$\frac{\phi''}{\phi'}-\frac{\phi'}{\phi} = -\frac{\omega'}{3+2\omega} \quad (2.16)$$

Equations (2.14) are exactly the same as in the Bianchi type I model in General Relativity. Only the constraint equation (2.15) is different. The solutions of the field equations are in the $\tilde{t}$ time the well-known Kasnerian solutions:

$$\tilde{a}=\tilde{t}^{p_1}, \tilde{b}=\tilde{t}^{p_2}, \tilde{c}=\tilde{t}^{p_3} \quad (2.17)$$

p_1, p_2, p_3 being the Kasner exponents with :

$$\sum p_i = 1 \quad (2.18)$$

With the constraint equation, we obtain:

$$\sum p_i^2 = 1-2\phi_0^{-2} \quad (2.19)$$

ϕ_0 being the integration constant of the scalar field. Hence, for all coupling constant $\omega(\phi)$, in the conformal frame, there will always be one negative Kasner exponent or three positive Kasner exponents and then two or three decreasing metric functions. In the $\tilde{\tau}$ time, the solutions of (2.14) are:

$$\begin{aligned}\tilde{a} &= e^{\alpha_1\tilde{\tau}+\alpha_0}\\ \tilde{b} &= e^{\beta_1\tilde{\tau}+\beta_0}\\ \tilde{c} &= e^{\gamma_1\tilde{\tau}+\gamma_0}\end{aligned} \quad (2.20)$$

where α_i, β_i, γ_i are integration constants. We integrate the Klein-Gordon equation to obtain the important equation:

$$\tilde{\phi}_0\phi'\phi^{-1}\sqrt{3+2\omega}=1 \quad (2.21)$$

$\tilde{\phi}_0$ being an integration constant (in fact $\tilde{\phi}_0 = A$). Hence, we deduce from the constraint equation that:

$$\alpha_1\beta_1+\alpha_1\gamma_1+\beta_1\gamma_1 = 1/4\tilde{\phi}_0^2, \forall\omega(\phi) \quad (2.22)$$

To find solutions to the field equations (2.7) we proceed as follow: first, we have to find solutions, for the scalar field, of the equations (2.9) and (2.21) so that we obtain respectively $\phi(\tau)$ and $\phi(\tilde{\tau})$. Second, we write $\phi(\tau)=\phi(\tilde{\tau})$ and reverse $\phi(\tilde{\tau})$ to find $\tilde{\tau}=\tilde{\tau}(\tau)$. Third, using (2.11), we write :

$$a=\tilde{a}(\tilde{\tau}(\tau))/\phi(\tau), b=\tilde{b}(\tilde{\tau}(\tau))/\phi(\tau), c=\tilde{c}(\tilde{\tau}(\tau))/\phi(\tau) \quad (2.23)$$

Let us examine what are the relations between the quantities in the τ time and in the t time. The amplitudes of the metric functions are the same in the both time since $a(\tau)=a(\tau(t))=a(t)$. The sign of the first derivatives are also the same : remember that $d\tau/dt = 1/abc$ is positive since the metric functions are positive-definite. Hence, τ is an increasing function of t and the sign of the first derivative of the metric functions will be the same in both τ time and t time. The sign of the second derivatives in the t time and τ time are different. If an overdot denotes differentiation with respect to t, the sign of $\ddot{a}$ will be that of $a''-a'(a'/a+b'/b+c'/c)$. We will study both the sign of a'' and $\ddot{a}$ in the applications of section 2.3. Of course, the amplitudes of the derivatives are different in the t and τ times. But we will not study them since we are mainly interested in their signs and therefore dynamical behaviour of the metric functions: whether they are increasing, decreasing or bouncing, and whether there is inflation.

Another difference between the two times comes from their asymptotic behaviours. For instance, the t time could diverge at a finite value of the τ time. It depends mainly on $dt/d\tau = abc = V$, where V is the volume of the Universe. In the cases we are going to study, the volume will always tend toward 0 or infinity (we will show it for the two first theories of section 2.3). Then, if $V \to 0$ when τ tends toward a constant or infinity, t tends toward a constant. If $V \to \infty$ when $\tau \to \infty$, $t \to \infty$. If $V \to \infty$ when τ tends toward a constant, t may tend toward infinity or a constant. In this last case, we need to integrate the volume abc to make the asymptotic behaviour of the cosmic time t precise. Unhappily, it will not be possible in the theories of section 2.3. We have studied the behaviour of the volume for the two first theories so that one can always get the asymptotic behaviour of $t(\tau)$ by using these rules (except the case $\tau \to cte$ and $V \to \infty$).

Concerning the presence of singularity, to ensure that a theory is non-singular, we will check that the Ricci curvature scalar R is finite. The Ricci scalar can be writen:

$$R = (abc)^{-2}\left[-\omega(\phi^{\cdot}\phi^{-1})^2 + 3\phi^{-1}\omega^{\cdot}\phi^{\cdot}(3+2\omega)^{-1}\right] \tag{2.24}$$

2.3 Non-singular and accelerated behaviours.

To simplify the study of the metric functions, we will consider in what follows only an increasing function of the scalar field, which means the only positive constants are A and $\tilde{\phi}_0$.

2.3.1 The case $3 + 2\omega = 2\beta(1 - \phi/\phi_c)^{-\alpha}$

We use the form for the coupling constant $3 + 2\omega = 2\beta(1 - \phi/\phi_c)^{-\alpha}$ where β is a positive constant, α, ϕ_c are constant. The case $\alpha = 0$ corresponds to Brans-Dicke theory and the case $\alpha = 1$ and $\beta = -1/2$ to Barker's theory [34]. Barrow showed in his paper [31] that the case $\alpha = 2$ is representative of the behaviour of other cases with $\alpha \neq 2$ in the neighbourhood of the singularity. Hence, we will consider only this case. From (2.9) we derive:

$$\phi(\tau) = \phi_c\left[1 - e^{-(\tau+\tau_0)/(A\sqrt{2\beta}\phi_c)}\right] \tag{2.25}$$

from (2.21) we deduce:

$$\phi(\tilde{\tau}) = \phi_c(1 + e^{-(\tilde{\tau}_0+\tilde{\phi}_0^{-1}\tilde{\tau})/\sqrt{2\beta}})^{-1} \tag{2.26}$$

Equating (2.25) and (2.26), we get:

$$\tilde{\tau} = \tilde{\phi}_0\sqrt{2\beta}ln\left[e^{(\tau+\tau_0)/(A\sqrt{2\beta}\phi_c)} - 1\right] - \tilde{\phi}_0\tilde{\tau}_0 \tag{2.27}$$

τ_0 being an integration constant. Hence, using (2.23), we write:

$$a(\tau) = \frac{e^{-\tilde{\phi}_0\tilde{\tau}_0\alpha_1+\alpha_0}}{\sqrt{\phi_c}}(e^{\frac{\tau+\tau_0}{A\sqrt{2\beta}\phi_c}} - 1)^{\sqrt{2\beta}\tilde{\phi}_0\alpha_1}(1 - e^{-\frac{\tau+\tau_0}{A\sqrt{2\beta}\phi_c}})^{-1/2} \tag{2.28}$$

and identical expressions for $b(\tau)$, $c(\tau)$ with β_0, β_1 and γ_0, γ_1 respectively. If we introduce:

$$u = (\tau+\tau_0)/(A\sqrt{2\beta}\phi_c),\, a_0 = e^{-\tilde{\phi}_0\tilde{\tau}_0\alpha_1+\alpha_0}/\sqrt{\phi_c} > 0,\, \alpha_1 = -\sqrt{2\beta}\tilde{\phi}_0\alpha_1 \tag{2.29}$$

the expression (2.28) becomes:

$$a(\tau) = a_0(e^u - 1)^{-a_1-1/2}e^{u/2} \tag{2.30}$$

u and the τ time vary in the same manner as long as A and ϕ_c are positive constants. The constraint equation (2.22) is rewritten as:

$$a_1b_1 + a_1c_1 + b_1c_1 = \frac{1}{2}\beta \tag{2.31}$$

The metric function will be real for positive u. One can show that there is no non-singular behaviour for this theory in an anisotropic Universe. The Ricci curvature can be written as:

$$R = (e^u - 1)^{1+2(a_1+b_1+c_1)}(3 - 2\beta e^{2u} - 24\beta^2e^{4u} + 24\beta^2e^{5u})(2a_0^2b_0^2c_0^2e^{3u})^{-1} \tag{2.32}$$

We check that conditions to get finite R for asymptotic times $(u \to 0, u \to \infty)$ are not compatible: for $u \to 0$ we need $a_1 + b_1 + c_1 > -1/2$ whereas for $u \to +\infty$, we need $a_1 + b_1 + c_1 < -3/2$. So there is always a singularity for the Ricci curvature at small or/and large times.

The first derivative of (2.30) shows that the metric function $a(\tau)$ will have a minimum for $u = -ln(-2a_1)$ and $a_1 \in]0, -1/2[$. For small u, we have $\phi \to 0$, $\omega \to \beta - 3/2$ and:

$$a \approx a_0(e^u - 1)^{-a_1-1/2} \tag{2.33}$$

Hence, if $a_1 < -3/2$, $da/d\tau$ and a tend to 0, if $a_1 \in [-3/2, -1/2]$, $da/d\tau$ tends to infinity and a tends to 0, if $a_1 > -1/2$, $da/d\tau$ and a tends respectively to $-\infty$ and $+\infty$. For large u, we have $\phi \to \phi_c$, $\omega \to +\infty$ if $\alpha > 0$ and:

$$a \approx a_0 e^{-a_1 u} \tag{2.34}$$

Hence, if $a_1 < 0$, $da/d\tau$ and a tend to infinity, if $a_1 > 0$, $da/d\tau$ and a tend to 0. We see that the form of the metric function depends only on the parameter a_1:

- If $a_1 < -3/2$, the metric function is increasing (fig 1).
- If $a_1 \in [-3/2, -1/2]$, it is increasing but with an inflexion point (fig 2). By studying the second derivative of $a(\tau)$, one can show that the condition to have an inflexion point is $a_1 \in [-3/2, -1/2]$. In the other cases, the second derivative is always positive and the dynamic is always accelerated. Lets note that it is not inflation since for that we must have $\ddot{a} > 0$ and not $a'' > 0$.
- If $a_1 \in [-1/2, 0]$, the metric function has a minimum. Hence, if a_1, b_1 and c_1 belong to $[-1/2, 0]$, all the metric functions have a bounce. However that does not mean that the Universe is non-singular since in this case the Ricci scalar become infinite for large τ.
- If $a_1 > 0$, the metric function is decreasing (fig 4).

Example of these four behaviours are illustrated on figures 1-4.

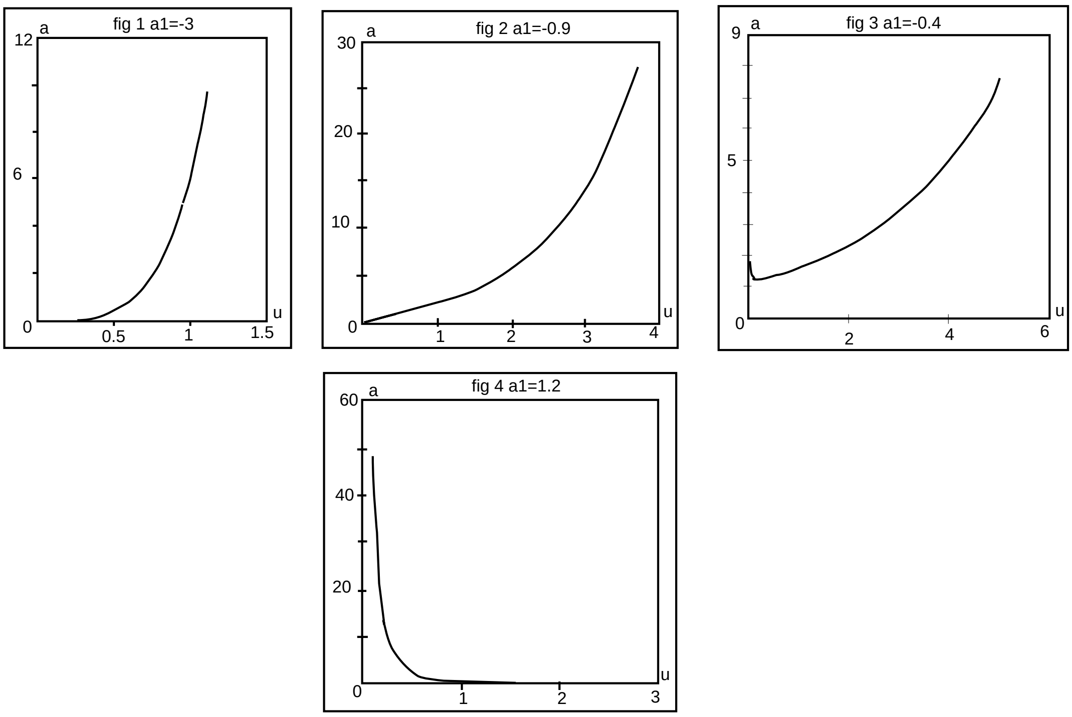

FIG. 2.1 – *Forms of the metric functions when* $3 + 2\omega = 2\beta(1 - \phi/\phi_c)^{-2}$.

Now we examine the sign of the second derivative of the metric function a in the t time so that we can detect inflation. It is the same as the quantity $-2a_1(b_1 + c_1)e^{2u} + (1 - b_1 - c_1)e^u - 1$ which is a second degree equation for e^u. One finds two roots: $e^{u_{1,2}} = (1 - b_1 - c_1 \pm \sqrt{\Delta})\left[4a_1(b_1 + c_1)\right]^{-1}$ with $\Delta = (b_1 + c_1 - 1)^2 - 8a_1(b_1 + c_1)$. If they are complex or inferior to 1, the sign of $\ddot{a}$ is the same as $-2a_1(b_1 + c_1)$. If they are superior to 1, there are two inflexion points: $\ddot{a}$ is first positive (negative), negative (positive) and then positive (negative) if $-2a_1(b_1 + c_1) > 0$ (respectively $-2a_1(b_1 + c_1) < 0$). For the same reasons, if one of the roots is not real or inferior to 1, there is one inflexion point and $\ddot{a}$ is first positive (negative) and then negative (positive) if $-2a_1(b_1 + c_1) > 0$ (respectively $-2a_1(b_1 + c_1) < 0$). Here, $\ddot{a} > 0$ can correspond to inflation when in the same time $\dot{a}$, or equivalently a', is positive. Hence, one see an example of kinetic inflation as described by Jana Levin in [27] and [28]. We remark also that inflation can end in a natural way.

If now we write the volume:

$$V = abc \tag{2.35}$$

For small (large) u, V vanishes if $a_1 + b_1 + c_1 < -3/2$ ($a_1 + b_1 + c_1 > 0$) else it tends toward infinity. Another interesting feature of this model is that for $\beta = 1/2$, we have $\omega \to -1$ for small value of u,

$\omega \to \infty$ and $\omega^{-3}(d\omega/d\phi) \to 0$ if $\alpha > 1/2$ for large value. That is the two value of the coupling constant that corresponds to String theory in the low-energy limit and to General Relativity (by General Relativity we means that the post-Newtonian parameters of General Relativity are recovered).

2.3.2 The case $3 + 2\omega = \phi_c^2\phi^{2m}$.

Now, we consider the following form of the coupling constant:

$$3 + 2\omega = \phi_c^2\phi^{2m} \tag{2.36}$$

where ϕ_c and m are real constants. Using the same process than before, from (2.9) we derive :

$$\phi(\tau) = [(m+1)/(A\phi_c)(\tau+\tau_0)]^{1/(m+1)} \tag{2.37}$$

and from (2.21) we get :

$$\phi(\tilde{\tau}) = \left[m(\tilde{\phi}_0\phi_c)^{-1}(\tilde{\tau}+\tilde{\tau}_0)\right]^{1/m} \tag{2.38}$$

Equating (2.37) and (2.38) we have:

$$\tilde{\tau} = \frac{\tilde{\phi}_0\phi_c}{m}\left[\frac{m+1}{A\phi_c}(\tau+\tau_0)\right]^{m/(m+1)} - \tilde{\tau}_0 \tag{2.39}$$

Then, with (2.23) we obtain :

$$a(\tau) = exp\{\frac{\alpha_1\tilde{\phi}_0\phi_c}{m}\left[\frac{m+1}{A\phi_c}(\tau+\tau_0)\right]^{m/(m+1)} - \alpha_1\tilde{\tau}_0 + \alpha_0\}\left[\frac{m+1}{A\phi_c}(\tau+\tau_0)\right]^{-1/2(m+1)} \tag{2.40}$$

We introduce the variables :

$$a_0 = e^{-\alpha_1\tilde{\tau}_0+\alpha_0},\, a_1 = \alpha_1\tilde{\phi}_0\phi_c,\, u = (\tau+\tau_0)/(A\phi_c) \tag{2.41}$$

and (2.40) becomes :

$$a = a_0 exp(a_1 m^{-1}\left[(m+1)u\right]^{m/(m+1)})\left[(m+1)u\right]^{-1/2(m+1)} \tag{2.42}$$

We get the same type of expressions for $b(\tau)$ and $c(\tau)$. From the constraint equation (2.22) we deduce:

$$a_1b_1 + a_1c_1 + b_1c_1 = \phi_c^2/4 \tag{2.43}$$

The expression (2.42) of the metric function shows that $(m+1)u$ must be positive. Hence, if $m > -1$, $u \in [0, +\infty[$ and if $m < -1$, $u \in]-\infty$,0]. u and the τ time vary in the same manner as long as A and $\phi_c = \sqrt{\phi_c^2}$ are two positive constants.

First, let us examine the Ricci scalar. It is written:

$$R = \left[(1+m)u\right]^{(1-2m)/(1+m)}[3 - \phi_c^2\left[(m+1)u\right]^{2m/(m+1)} + 6m\phi_c^4\left[(m+1)u\right]^{4m/(1+m)}]\left[2a_0^2b_0^2c_0^2(1+m)^2e^{2(a_1+b_1+c_1)[(m+1)u]^{m/(m+1)}/m}\right]^{-1} \tag{2.44}$$

Only if $m \in$ [0,1/2] and $a_1 + b_1 + c_1 > 0$, is the Ricci scalar always finite at both small and large times, avoiding the singularity. Now we examine the dynamic of a in the τ time. The first derivative of (2.42) vanishes for $u = (2a_1)^{-(m+1)/m}/(m+1)$ and hence, $a(\tau)$ has an extremum for this value that exists only if a_1 is positive. The asymptotic study of (2.42) when $u \to 0$ and $u \to \pm\infty$ gives the results summarised in table 1. We found eight different behaviours. The figures 5-12 show an example of each of them. To summarise the main characteristics of each case in the τ time:

- For $a_1 < 0$, the metric function is always decreasing and has an inflexion point when $m < -3/2$.
- For $a_1 > 0$, the metric function has a minimum if $m > 0$ and a maximum if $m < 0$. Hence, only the case where a_1,b_1,c_1 and m are positive, gives birth to a "bounce" Universe. It avoids the singularity if $m \in$ [0,1/2] and $a_1 + b_1 + c_1 > 0$ and will be today in expansion in all directions of space.

	$a_1 < 0$	$a_1 > 0$
m¿0	$u \to 0^+$,$a \to +\infty$,$a' \to -\infty$ $u \to +\infty$,$a \to 0^+$,$a' \to 0$	$u \to 0^+$,$a \to +\infty$,$a' \to -\infty$ $u \to +\infty$,$a \to +\infty$,$a' \to +\infty$
$m \in [-1,0]$	$u \to 0^+$,$a \to +\infty$,$a' \to -\infty$ $u \to +\infty$,$a \to 0^+$,$a' \to 0$	$u \to 0^+$,$a \to 0$,$a' \to +\infty$ $u \to +\infty$,$a \to 0$,$a' \to 0$
$m \in [-3/2,-1]$	$u \to 0^-$,$a \to 0^+$,$a' \to 0$ $u \to -\infty$,$a \to +\infty$,$a' \to -\infty$	$u \to 0^-$,$a \to 0$,$a' \to -\infty$ $u \to -\infty$,$a \to 0$,$a' \to 0$
$m < -3/2$	$u \to 0^-$,$a \to 0$,$a' \to 0$ $u \to -\infty$,$a \to +\infty$,$a' \to -\infty$	$u \to 0^-$,$a \to 0$,$a' \to -\infty$ $u \to -\infty$,$a \to 0$,$a' \to 0$

TAB. 2.1 – *The eight different asymptotic behaviours of the metric function when $3 + 2\omega = \phi_c^2\phi^{2m}$. The asymptotic amplitudes of a are the same in t and τ time. That is not the case for the amplitudes of the first derivatives. We do not examine the asymptotic behaviour of the amplitudes of $\dot{a}$ since we are mainly interested by the study of the exact solutions in the τ time and, in a general maner, by the signs of a', a'' and $\ddot{a}$. But this is always possible by calculating $\dot{a} = a'(abc)^{-1}$.*

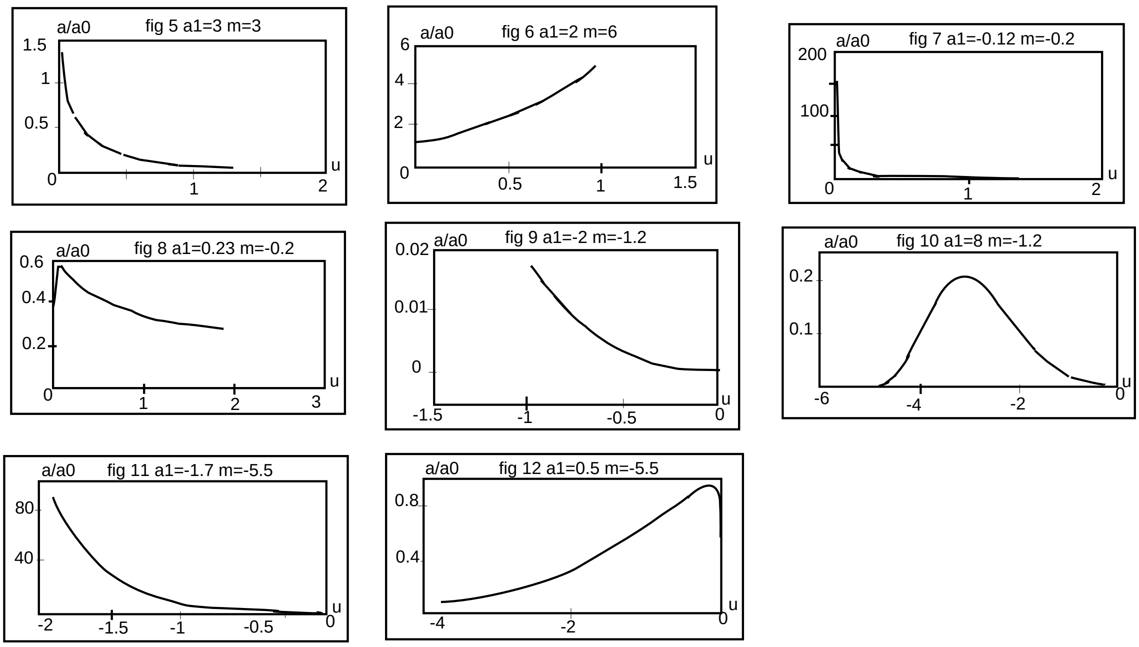

FIG. 2.2 – *Forms of the metric functions when $3 + 2\omega = \phi_c^2\phi^{2m}$.*

If we define the volume V by $V = abc$, then it tends to vanish for small u if $m/(m+1) < 0$ and $m(a_1 + b_1 + c_1) < 0$ or $m/(m+1) > 0$ and $-3/\left[2(m+1)\right] > 0$. It becomes infinite if $m/(m+1) < 0$ and $m(a_1 + b_1 + c_1) > 0$ or $m/(m+1) > 0$ and $-3/\left[2(m+1)\right] < 0$. For large u, it tends to vanish if $m/(m+1) > 0$ and $m(a_1 + b_1 + c_1) < 0$ or $m/(m+1) < 0$ and $-3/\left[2(m+1)\right] < 0$. It becomes infinite if $m/(m+1) > 0$ and $m(a_1 + b_1 + c_1) > 0$ or $m/(m+1) < 0$ and $-3/\left[2(m+1)\right] > 0$.

By examining the sign of a'', we can conclude that the dynamic of the metric function will always be accelerated (recall again that it is not inflation since it does not mean that $\ddot{a} > 0$) if $m > 1/2$ or $m \in [-3/2,1/2]$ and $a_1 < 0$. If $m < -3/2$ the dynamic is first accelerated and then decelerated. The same thing happens when $m \in [0,1/2]$ and $a_1 > 0$ whereas for $m \in [-3/2,0]$ and $a_1 > 0$, the metric accelerates again.

We complete this study by examining the sign of the second derivative in the t time. It is the same as $m + (b_1 + c_1)\left[\left[(m+1)u\right]^{m/(m+1)} - 2a_1\left[(m+1)u\right]^{2m/(m+1)}\right]$. This is a second degree equation for $\left[(m+1)u\right]^{m/(m+1)}$. The two roots are

$$u_{1,2} = (m+1)^{-1}\left[(b_1 + c_1 \pm \sqrt{\Delta})(4a_1(b_1 + c_1))^{-1}\right]^{(m+1)/m} \tag{2.45}$$

with $\Delta = (b_1 + c_1)(8a_1 m + b_1 + c_1)$. If $u_{1,2}$ are not real, the sign of $\ddot{a}$ is the one of $-2a_1(b_1 + c_1)$. When the two roots are real, they always belong to the interval where u varies since their sign is the same as $m+1$.

Then $\ddot{a}$ has the same sign as $-2a_1(b_1+c_1)$ if u is out of $[u_1,u_2]$ or the opposite sign if $u \in [u_1,u_2]$. There are two inflexion points. Hence, we get the same type of behaviour for $\ddot{a}$ as in the previous subsection. In the same manner, if only one root is real, the dynamic of a will be accelerated and then decelerated or vice-versa depending on the sign of $-2a_1(b_1+c_1)$. So, there is one inflexion point. For this theory also, inflation can end naturally.

Concerning the coupling constant, we have for $m+1>0$: when $\tau \to +\infty$, $\phi \to +\infty$, $\omega \to \phi_c^2\phi^{2m}/2 \to +\infty$ if $m>0$ and $\omega \to -3/2$ if $m \in [-1,0]$. When $\tau \to \tau_0$, $\phi \to 0$, $\omega \to \phi_c^2\phi^{2m}/2 \to +\infty$ if $m \in [-1,0]$ and $\omega \to -3/2$ if $m>0$. Considering these last remarks and the relation (3), one can deduce that the asymptotic behaviours of the metric functions when $\phi \to 0$, $\omega \to \phi_c^2\phi^{2m}/2 \to +\infty$ and $m \in [-1,0]$ are the same as in the cases of a coupling function of type $f(\Phi)=f_0e^{n\Phi}$ when $\phi_c^2=n^{-2}$ and $m=-1/2$ and $f(\Phi)=(f_0\Phi+f_1)^n$ when $\phi_c^2=(f_0n)^{-2}$ and $2m=(2-n)/n$ with $n \notin [0,2]$. Moreover, the asymptotic behaviour of the metric functions when $\phi \to +\infty$, $\omega \to \phi_c^2\phi^{2m}/2 \to +\infty$ and $m>0$ are the same as in the previous case but with $n \in [0,2]$.
Hence the study of the metric functions when $3+2\omega=\phi_c^2\phi^{2m}$, give us information on the asymptotic behaviours of two different couplings $f(\Phi)$, that is $f(\Phi)=(f_0\Phi+f_1)^n$ and $f(\Phi)=f_0e^{n\Phi}$. For the first of these functions, the minimally coupled theory is obtained for $f_0=0$ and $f_1^n=1/2$, whereas the induced gravity is obtained for $f_1=0$, $f_0=\sqrt{\epsilon/2}$ and $n=2$. We note that the study of one coupling constant $\omega(\phi)$ permit us to get informations on two types of coupling $f(\Phi)$ because $\omega(\phi)$ and $f(\Phi)$ are linked by the differential equation (2.4). Hence to one type of function ω, having one or several free parameters, can correspond more than one type of functions f. What we say above comes from the fact that to a power or exponential law for $f(\Phi)$ correspond only a power law for $\omega(\phi)$.

2.3.3 The case $3+2\omega=e^{2\phi_c\phi}$.

We take the form $3+2\omega=e^{2\phi_c\phi}$, ϕ_c being a real constant. This is an interesting case because, as in the subsection 2.3.1, when the scalar field vanishes, the coupling constant tends towards -1, which is the low limit of the string theory, whereas when it becomes infinite, the coupling constant tends towards infinity and the theory towards General Relativity if $\phi_c>0$.

Here, we can not integrate equation (2.21) in a closed convenient form. We rewrite the equations (2.9) and (2.21) in the following form:

$$H(\phi)=\tau=\int Ae^{\phi_c\phi}d\phi-\tau_0=A\phi_c^{-1}e^{\phi_c\phi}-\tau_0 \tag{2.46}$$

$$G(\phi)=\tilde{\tau}=\int \tilde{\phi}_0e^{\phi_c\phi}\phi^{-1}d\phi-\tilde{\tau}_0 \tag{2.47}$$

That means we have $\phi(\tau)=H^{(-1)}(\tau)$ and $\phi(\tilde{\tau})=G^{(-1)}(\tilde{\tau})$. By equalling these last two expressions and reversing (2.46), we get :

$$\tilde{\tau}=G(H^{(-1)})=G(\phi)=G(\phi_c^{-1}ln\left[\phi_cA^{-1}(\tau+\tau_0)\right]) \tag{2.48}$$

With (2.23), we can easily obtain the metric functions :

$$a=e^{\alpha_1G(\phi_c^{-1}ln\left[\phi_cA^{-1}(\tau+\tau_0)\right])+\alpha_0}/\sqrt{(A\phi_c)^{-1}ln\left[\phi_c(\tau+\tau_0)\right]} \tag{2.49}$$

and the same form for $b(\tau)$ and $c(\tau)$ with their integration constants. The reality conditions for the metric functions will be $\phi_cA^{-1}(\tau+\tau_0)>0$ and $\phi_c^{-1}ln\left[\phi_cA^{-1}(\tau+\tau_0)\right]>0$.

Hence, if $\phi_c<0$, the metric function will be real if $\tau \in \left]A\phi_c^{-1}-\tau_0,-\tau_0\right[$, and if $\phi_c>0$, we will have $\tau \in]A\phi_c-\tau_0,+\infty[$. The first derivative of (2.49) will be of the sign of $\alpha_1\tilde{\phi}_0\phi_cA^{-1}(\tau+\tau_0)-1/2$. For all value of ϕ_c, when $\tau=A(2\alpha_1\phi_c\tilde{\phi}_0)^{-1}-\tau_0$, $da/d\tau$ vanishes in the following cases:

- when $\tau \in \left]A\phi_c^{-1}-\tau_0,-\tau_0\right[$, that means $\phi_c<0$, if $2\alpha_1\tilde{\phi}_0>1$,

- when $\tau \in \left]A\phi_c^{-1}-\tau_0,+\infty\right[$, that means $\phi_c>0$, if $2\alpha_1\tilde{\phi}_0 \in [0,1]$. From these results and after a numerical study we can write that :

- If $\phi_c<0$, $\tau \in]A\phi_c-\tau_0,-\tau_0[$:
  - If $\alpha_1<(2\tilde{\phi}_0)^{-1}$, the metric function is decreasing and tends to infinity, in a positive manner when $\tau \to A\phi_c^{-1}-\tau_0$, and to zero when $\tau \to -\tau_0$.

  - If $\alpha_1 > (2\tilde{\phi}_0)^{-1}$, the metric function tends to zero for these two values of τ and has a maximum . So, if the three integration constants α_1, β_1, γ_1 of each of the metric functions are such that they are all superior to $(2\tilde{\phi}_0)^{-1}$, we have a close Universe (for the time) which exists during a finite time in the τ-time. Since $dt/d\tau = abc$, this quantity vanishes in $\tau = A\phi_c^{-1} - \tau_0$ and $\tau = \tau_0$ and then $t(\tau)$ stays finite for these two values and the Universe also exists during a finite t time.
- If $\phi_c > 0$, $\tau \in]A\phi_c - \tau_0, +\infty[$:
  - If $\alpha_1 < 0$, the metric function decreases from infinity to zero.
  - If $\alpha_1 \in \left[0,(2\tilde{\phi}_0)^{-1}\right]$, the metric function has a minimum and tends to $+\infty$ when τ tends to $A\phi_c^{-1} - \tau_0$ or $+\infty$. If the three integration constants α_1, β_1, γ_1 are all in the same interval, the Universe will have a bounce since each metric function has a minimum.
  - If $\alpha_1 > (2\tilde{\phi}_0)^{-1}$, the metric function is increasing from zero to infinity with an infinite slope.

2.4 Conclusion

In the conformal frame, the scalar field is minimally coupled. Hence, the spatial components of the field equations are exactly the same as in General Relativity and their solutions for the Bianchi type I model are the kasnerian solutions [29]. The Klein-Gordon equation and the constraint equation, that are different from General Relativity, impose that the sum of the square of the Kasner exponents is always inferior to unity. Their sum is equal to one. Hence, there are always two or three positive Kasner exponents.

To express the metric function in the Brans-Dicke frame, we have equated the expressions of the scalar field in both Brans-Dicke and conformal frames and then deduced the time $\tilde{\tau}$ of the conformal frame as a function of the time τ of the Brans-Dicke frame. Then it is easy to find the form of the metric functions in this last frame. The amplitude of the metric functions and the sign of their first derivative in the τ time of the Brans-Dicke frame are the same as in the t time. This is not the case for the second derivative of the metric functions.

We have studied three forms of the coupling constant $\omega(\phi)$ and found solutions for which the Universe could avoid the singularity. We have also detected kinetic inflation for the two first examples and notice that, under some conditions, it can end naturally. For small or large value of the τ time, the coupling constant can become infinite or constant. It is always interesting to find classes of coupling constant for which it tends naturally toward -1 or infinite for small or large value of τ because such a class of theories tends respectively toward string theory in the low-energy limit and General Relativity. It seems to be true in the special case $3 + 2\omega = (1 - \phi/\phi_c)^{-2}$ and for $3 + 2\omega = e^{2\phi_c\phi}$.

Chapitre 3

Solutions exactes pour le modèle de Bianchi de type I: en se donnant des fonctions du temps propre(1 article)

Dans l'article qui suit, nous allons exprimer sous forme de quadratures la solution des équations de champs de la théorie tenseur-scalaire non minimalement couplée et massive définie par

$$L = G^{-1}R - \frac{\omega}{\phi}\phi_{,\mu}\phi^{,\mu} - U$$

en fonction du 3-volume V de l'Univers et du potentiel U du champ scalaire pour le modèle de Bianchi de type I. Contrairement au chapitre précédent, le champ scalaire ϕ est désormais massif et la fonction de gravitation est une fonction inconnue de ϕ. Il y a donc trois fonctions indéterminées du champ scalaire, G, ω et U. Nous pourrions à nouveaux choisir la forme de ces fonctions de ϕ et tenter de résoudre les équations. Cependant il est difficile de faire des choix physiquement justifiés en dehors de théories "classiques" comme par exemple la théorie de Brans-Dicke avec une constante cosmologique. De plus la résolution directe des équations de champs comme dans le chapitre précédent est la plupart du temps impossible, en partie en cause de la présence du potentiel qui empêche le calcul de ϕ.
En revanche, exprimer les solutions exactes en fonction du 3-volume de l'Univers et du potentiel offre une alternative à ces problèmes. D'une part, cette résolution est mathématiquement réalisable à l'aide de quadrature. D'autre part, il existe des formes très générales de V et de U qu'il est physiquement justifié d'étudier. Entre autre, nous examinerons les théories telles que ces deux quantités tendent vers des puissances ou des exponentielles du temps propre. On retrouve ces comportements pour de nombreuses théories et, bien que nous considèrerons nos solutions comme des solutions exactes, leurs résultats pourront également être étendus à n'importe quelle théorie tenseur-scalaire dont le 3-volume et le potentiel convergent suffisamment vite vers ces fonctions. Nous verrons l'importance de ces comportements lorsque nous étudierons l'isotropisation des modèles de Bianchi.
Cette méthode est intéressante lorsque l'on a une idée du comportement asymptotique du 3-volume et du potentiel en fonction du temps propre mais elle ne permet pas de contraindre les théories tenseur-scalaires afin qu'elles produisent des comportements plus généraux que ceux fixés par la donnée de quelques fonctions du temps propre. Entre autre dans les chapitres suivants, nous verrons qu'il est possible d'obtenir des contraintes sur les fonctions du champ scalaire telles que l'Univers soit asymptotiquement en expansion accélérée, ceci sans préciser aucune fonction de t mais au prix d'une simplification du Lagrangien ci-dessus.

Exact solutions of the Hyperextended Scalar Tensor theory with potential in the Bianchi type I model.

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Abstract

The Hyperextended Scalar Tensor theory with a potential is defined by three free functions: the gravitational function, the Brans-Dicke coupling function and the potential. Starting from the expression of the 3-volume and the potential as function of the proper time, we determine the exact solutions of this theory. We study two important cases corresponding to power and exponential laws for the 3-volume and the potential.

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3.1 Introduction

In this work, we study the Hyperextended Scalar Tensor theory (HST) with a potential for the Bianchi type I model. We determine, by help of quadratures, the exact solutions of this theory as function of the 3-volume and the potential. We then study the case for which these 2 functions are power or exponential laws of the proper time.

The simplest scalar tensor theories is the Brans-Dicke theory studied in the sixties by Brans and Dicke [7]. One of their goals was to integrate the Mach's principle in a gravitational relativistic theory. Since the eighties, other justifications have been given to take into account scalar fields in gravitational theories. They are issued from inflation and particles physics theories, i.e. the unification models. Their low energy limit can be described by scalar tensor theories. As instance, it is the case for supergravity theory or higher dimensional theories. Large classes of scalar tensor theories belong to the HST [35]. Its Lagrangian is written as this of the Brans-Dicke one but the coupling constant is replaced by a function of the scalar field and the gravitational function , ϕ in the Brans-Dicke theory, by any function $G^{-1}(\phi)$. In this paper, we will also consider a potential $U(\phi)$ which is predicted by particles physics for the early epochs. Moreover, recent studies on the type I supernovae [9, 10] could confirm the presence of a cosmological constant, which would be the remainder of the potential for late time epochs. We will not consider other type of matter as perfect fluids, thus assuming a Universe dominated by the scalar field. Such phases for the Universe are relevant: near the singularity, perfect fluid are often negligible [36]. Furthermore, it is sometimes considered that scalar fields could be responsible for a large part of the dark matter which could be the dominant matter of our Universe.

Lets justify the geometrical framework of this paper. It is well known that large scale structures we observe could not exist if the Universe has always respected the cosmological principle. Thus it seems reasonable to consider other models such as the homogeneous ones: these are the Bianchi models. Some of them have the interesting property to isotropize toward an FLRW model: the Bianchi type V model can approach the open FLRW one, the Bianchi type IX model can tend toward the closed FLRW one and the Bianchi type I model toward the flat FLRW one. Recent observations from Boomerang [37] seem favour closed models. However, from the point of view of inflationary models, the flat model is the most studied and is usually preferred to other ones. Hence, it is difficult today to decide what is the best model to describe our Universe and we will choose to study the Bianchi type I one.

Now, we give a more accurate description of the functions characterising the HST. When a potential is present, it is defined by three free functions: the function G playing the role of a variable gravitational function, the Brans-Dicke coupling function describing the coupling between the scalar field and the metric, and last the potential U. To find exact solutions, most of times one choose $G(\phi)$, $\omega(\phi)$ and $U(\phi)$ and determine the form of the metric functions. However, other methods exist to achieve this goal which have been mainly applied to the FLRW models. As instance, in the fine turning potential method [34, 38, 39], one first choose the form of the metric functions and then look for the form of the potential. In this paper, we will assume that the forms of the potential and the 3-volume of the Universe are known functions of the proper time. Despite interesting works to determine the form of the potential from the observations [40, 41], there is no method today to predict it for the HST. The second quantity is related to the isotropic part e^{Ω} of the metric [1] or the scale factor of the FLRW models. We will use theoretical considerations to choose its form as a function of the proper time. From these two quantities, we can get the exact forms of the gravitational function $G(t)$ and the anisotropic part of the metric. Moreover, if we choose a form for $\phi(t)$, we obtain $\omega(t)$ and then the theory and the solution of the field equations are fully determined.

Our motivations are the following:

– To find the exact solutions of the field equations when we know the isotropic part of the metric and the potential as functions of the proper time. This is a mathematical motivation corresponding to an extension of the fine turning potential.

– To study the dynamical properties of the Universe (isotropisation, inflation...) for special forms of e^{Ω} and U, i.e. power law and exponential law of the proper time in this paper. This deserves physical motivation since most of this results could be extendable to any function e^{Ω} and U asymptotically tending toward these special forms. Thus the scalar tensor theories whose the isotropic part and the potential can be asymptotically written as power series of t or exponential of t will be concerned by these results.

This paper is organised as follows: in the section 3.2 we write the field equations and give their exact solutions. In section 3.3, we look for the properties of the models defined by ($e^{\Omega} = t^m$,$U = t^n$) and ($e^{\Omega} = e^{mt}$,$U = e^{nt}$). We conclude in section 3.4.

3.2 Exact solution of the field equations of the HST with potential in the Bianchi type I model

We use the following form of the metric:

$$ds^2 = -dt^2 + e^{2\alpha}(\omega^1)^2 + e^{2\beta}(\omega^2)^2 + e^{2\gamma}(\omega^3)^2 \tag{3.1}$$

The ω_i are the 1-forms specifying the Bianchi type I model. We introduce the parameterisation:

$$\begin{aligned} \alpha &= \Omega + \beta_+ \\ \beta &= \Omega + \beta_- \\ \gamma &= \Omega - \beta_+ - \beta_- \end{aligned}$$

It is similar to this of Misner [24]. The function Ω stands for the isotropic part of the metric and the functions $\beta_\pm$ describe the anisotropic part. The isotropic part is linked to the 3-volume V of the Universe by the relation $V = e^{3\Omega}$. The action of the HST is written:

$$S = \int (G^{-1}R - \frac{\omega}{\phi}\phi_{,\mu}\phi^{,\mu} - U)\sqrt{g}dx^4 \tag{3.2}$$

ϕ is the scalar field, U the potential, ω the Brans-Dicke coupling function and G the gravitational function. Lets justify the study of this action. Since in this paper we will have no need to assume any relation between the scalar field and the Brans-Dicke coupling function, we could use the action of the Generalised Scalar Tensor theory (GST) which has the same form as (3.2) but with $G^{-1} = \phi$. However we do not want to impose any relation between G and the scalar field since $G^{-1} = \phi$ is not the only form of the gravitation function in the literature. As instance String theory at low energy is defined by $G^{-1} = e^{\phi}$ and important studies have been made with gravitational function writing as $G^{-1} = \phi^2 + constant$. Hence, although we have no need for this in this paper, we will consider a general form for G. Lets underline that some HST theories can not be cast into a GST when the function G is not invertible although this change of variable is then singular and could be an indication for mathematical inconsistency [2].

The action (3.2) could also be equivalent to the General Relativity plus a minimally coupled scalar field if we redefine the metric functions as $\bar{g}_{\mu\nu} = G^{-1}g_{\mu\nu}$ [3]. We get then the so-called Einstein frame and the metric (3.1) is the so-called Brans-Dicke frame. We have chosen to work with the last metric since the results

1. We have then that the 3-volume is equal to $e^{3\Omega}$.

2. I thank one of the referees for having clarified this point.

3. Lets note that some results for the General Relativity with a minimally coupled scalar field can be get from these of the HST by putting $G^{-1} = 1$. But for obvious reasons it is not so simple to get results for the HST from these of the General Relativity with a minimally coupled scalar field.

we would get in the Einstein frame would not have been equivalent to these of the Brans-Dicke frame: as shown, as instance, in [42], and contrary to what happens when we do a "simple" scalar field redefinition, the conditions for the isotropisation in both frames are not always the same. The same conclusion arises for the presence or not of inflation. Thus, the results get in the Brans-Dicke frame for the HST will not be equivalent to these found in the Einstein frame or/and for General Relativity with minimally coupled scalar field, it is rather a generalisation. Moreover, the Brans-Dicke frame is generally assumed to be the physical one, although this point can be subject to discussion. One could also ask why we have not first studied the Einstein frame and then extended our results to the Brans-Dicke one. However, to proceed we would have to integrate $G^{-1}(\bar{t})$, which is not always workable.

We get the field equations by varying the action with respect to the metric functions. In the τ time defined by $dt = V d\tau$ we obtain:

$$\alpha'' + \alpha' G G^{-1\prime} + 1/2 G G^{-1\prime\prime} - 1/2 G V^2 U = 0 \tag{3.3}$$

and similar equations for β and γ. The prime stands for the derivative with respect to τ. For the constraint, we get

$$\alpha'\beta' + \alpha'\gamma' + \beta'\gamma' + GG^{-1\prime}V'V^{-1} - 1/2UGV^2 - 1/2\omega G\phi'^2\phi^{-1} = 0 \tag{3.4}$$

By adding the three spatial components, we find a differential equation for the 3-volume:

$$V''V^{-1}G^{-1\prime} - V'^2V^{-1}G^{-1} + V'V^{-1}G^{-1\prime} + 3/2G^{-1\prime\prime} - 3/2UV^2 = 0 \tag{3.5}$$

If we use equation (3.5) to express UV^2 and introduce this quantity in (3.3), we have for $\beta_\pm$:

$$\beta_\pm = \beta_{\pm 0} \int G e^{-3\Omega} dt + \beta_{\pm 1} \tag{3.6}$$

$\beta_{\pm 0}$ and $\beta_{\pm 1}$ are integration constants. Thus, the Universe isotropize when $t \to \infty$ if $\int G e^{-3\Omega} dt$ tends toward a constant. Now, we want to evaluate the gravitational function G depending on Ω and U. We find with help of (3.5):

$$G^{-1} = e^{-2\Omega}\left[\int \frac{\int U e^{3\Omega} dt + G_0}{e^{\Omega}} dt + G_1\right] \tag{3.7}$$

G_0 and G_1 are constants. We can make two remarks:

– We have completely determined $G(t)$ and $\beta_\pm(t)$ as functions of $\Omega(t)$ and $U(t)$. The solutions of the spatial field equations are independent on the form of the scalar field and the Brans-Dicke coupling function since they depend only on the gravitational coupling function which is expressed as a function of the proper time and not of the scalar field.

– Moreover, we can write G^{-1} as:

$$G^{-1} = g_1(\Omega) + g_2(\Omega, U) \tag{3.8}$$

Then, writing $U = \sum_n U_n$, we see that $G^{-1}(\Omega, \sum_n U_n) = g_1(\Omega) + \sum_n g_2(\Omega, U_n)$. Thus, from the solution of the field equations for n potentials, we should be able to determine the solution for their sum. As instance, if we know then for a potential writing as t^n, we will be able to deduce the solution for any potential writing as a power law series.

Now, let's express the Brans-Dicke coupling function as function of Ω, U and ϕ. Using the constraint equation and (3.6), we get:

$$\omega = 2G^{-1}\dot{\phi}^{-2}\phi\left[3\dot{\Omega}^2 - G^2e^{-6\Omega}(\beta_{+0}^2 + \beta_{-0}^2 + \beta_{+0}\beta_{-0}) + 3G\dot{G^{-1}}\dot{\Omega} - 1/2GU\right] \tag{3.9}$$

The Brans-Dicke coupling function is then fully defined by $\Omega(t)$, $U(t)$ and $\phi(t)$. It exists the same type of linearity relation between ω and U as for G. We have:

$$\omega(\phi, \Omega, \sum_n U_n) = \omega_1(\phi, \Omega) + \sum_n \omega_2(\phi, \Omega, U_n) \tag{3.10}$$

In the next section, we study two classes of models for which the isotropic part of the metric and the potential are power or exponential laws of the proper time. For the clarity of the discussion, we will assume that their solutions are defined in $t \to +\infty$, which represents the late times epoch.

3.3 Application

3.3.1 Power laws

We choose power law forms for the isotropic part of the metric and the potential:

$$\begin{aligned} e^{\Omega} &= t^m \\ U &= U_0 t^n \end{aligned} \tag{3.11}$$

U_0 is a constant. When the Universe isotropize, the metric functions tends toward e^{Ω} which can be then compared to the scale factor of the FLRW models. In the flat isotropic models, the scale factor often takes power law forms as for General Relativity with perfect fluid. It is also an important form for the inflation, which received the name of polynomial inflation. In a general way, the association of the forms (3.11) is physically meaningful for many theories studied in the FLRW models. As instance, such forms for the scale factor and the potential have been found in [43] where a superpotential is considered. This is also asymptotically the case in [44] where conformal scalar field cosmologies are examined and in general for any forms of e^{Ω} and U which can be asymptotically developed as power law series. Thus, the results which follow could apply to large class of scalar tensor theories. Additional reasons will be given in the next section.

1. ***Gravitational function***
   From (3.7), we get:
   $$G^{-1} = C_1 t^{2+n} + C_2 t^{-2m} + C_3 t^{-3m+1} \tag{3.12}$$
   C_i are integration constants. From (3.12), we see that we can not choose m and n such that G be constant unlike asymptotically. Thus, General Relativity does not belong to the class of theories defined by these forms of Ω and U. When the Universe isotropizes, it will be in expansion if $m > 0$ and will undergo inflation if $m > 1$. The potential will tend naturally toward zero if $n < 0$. From (3.6) we deduce that isotropisation will happen at late times if $m > 1$ or $3m + n > -1$. In the first case, the Universe will be necessarily inflationary for this period. In the last case, inflation will go with isotropisation if $n < -4$. Thus, we can get an isotropic Universe without inflation. Consequently, if $m > 1$ or $3m + n > -1$, the Universe isotropises and the power law t^m represents a late times attractor for the metric functions.
   Power laws of the proper time for the gravitational function play an important role toward the literature. Milne, in the thirties, studied the case $G = t$ and Dirac, in the framework of the "Large Number hypothesis", proposed $G = t^{-1}$ [6]. More recently, in [45] a study of the Newtonian cosmologies with polynomial laws for G and perfect fluid ($p = (\gamma - 1)\rho$) in the isotropic models is made. It is shown that for $G = t^p$, inflation is present when $3\gamma > 2$ and do not depends on the variation of G. In this work, a condition to get asymptotically inflation is $m > 1$. In this case, $G \to t^{-(2+n)}$ if $n > -4$. Then, it shows that inflation in the class of models we are studying, is also asymptotically independent on the variation of the gravitational function as in [45] if the potential is larger than t^{-4}.
2. ***Applications***
   In this part, we examine several known types of Universes which are late times attractor when isotropisation arises.
   – Coasting Universe
     We choose $m = 1$. Then, the Universe isotropizes at late times and tends toward a coasting Universe, i.e. the metric functions tend toward t if $n > -4$. We calculate the exact solutions of the field equations. The anisotropic part of the metric is written:
     $$\beta_{\pm} = -\beta_{\pm 0} \frac{\ln\left[(C_2 + C_3)t^{-4-n} + C_1\right]}{(4+n)(C_2 + C_3)} + \beta_{\pm 1} \tag{3.13}$$
     Coasting Universe has been previously studied in [46]. In this paper, a Brans-Dicke model with a perfect fluid and a power law potential in a FLRW model was considered. For open, closed or flat models, they found linear expansion of the scale factor with a potential decreasing inversely with the square of time. An important characteristic of an isotropic coasting cosmologies is that the age of the Universe is not in conflict with the observations. So, the age problem is absent for this type of dynamical behaviour for the metric functions.
   – Cosmological constant
     We examine the case $m = 1/2$ and $n = 0$. The potential is then a cosmological constant. At late

times, the Universe isotropizes and the metric functions tend toward $t^{1/2}$ which is the form of the scale factor in an FLRW model for General Relativity when Universe is radiation dominated. This theory could thus build a bridge between an anisotropic Universe dominated by the scalar field and a flat relativistic and isotropic Universe dominated by the radiation. The gravitational function will behave asymptotically as t^{-2} and then will tend to vanish at late times. The exact solution for the metric functions can be found. The anisotropic part of the metric is written as:

$$\beta_\pm = \beta_{\pm 0} \sum_{i=1}^{6} \frac{\ln(\sqrt{t} + s_i)}{C_3 + 6C_1 s_i^5} + \beta_{\pm 1} \tag{3.14}$$

The s_i are the i^{th} roots of the equation $C_2 + C_3 s + C_1 s^6 = 0$. A cosmological constant is equivalent to consider an equation of state for the stiff fluid ($p = -\rho$). This situation has been studied in [45] where $G = t^p$. In this last paper it has been noticed that the asymptotical behaviour of the scale factor was determined by the value of p, the power of the gravitational function. Here, for the class of theories defined by (3.11), whatever $m > 0$, i.e. an asymptotically isotropic and expanding Universe, the gravitational function always behaves as t^{-2} at late times in presence of a cosmological function. The behaviour of G is thus independent on the value of m.

– Gravitational constant
  Another interesting case corresponds to $m = 1/2$ and $n = -2$. The anisotropic part of the metric functions is written:

$$\beta_\pm = \beta_{\pm 0} 4 \arctan\left(\frac{C_3 + 2C_1\sqrt{t}}{(4C_1C_2 - C_3^2)^{1/2}}\right)(4C_1C_2 - C_3^2)^{-1/2} + \beta_{\pm 1} \tag{3.15}$$

  Then, the potential tends to vanish at late times. The gravitational function tends asymptotically toward the constant C_1. One more times, this theory connects an anisotropic Universe to an isotropic one behaving dynamically as if radiation was present, but here the gravitational function tends toward a constant. In a general manner, when $m > 1/3$, the case $n = -2$ is the only one giving birth to an asymptotically non-vanishing gravitational constant. Since, recent observations suggest that our present Universe could undergo inflation, which means $m > 1$, this remark underline the importance of a potential behaving like t^{-2} at late times if we assume a gravitational constant for this epoch and a power law behaviour for the scale factor. This type of potential has been studied in [47, 48] and particularly in [46] where it arises naturally when one use a scalar field behaving as a power law type of the proper time.
– Static Universe
  For $m = 0$ and $n > -1$, the Universe will isotropize toward a static behaviour at late times. If moreover we require that the potential be decreasing, we need $n \in [-1{,}0]$. The anisotropic part of the metric takes the form of a hypergeometric function multiplied by t. Static phases for the Universe are interesting since they can help to solve the age problem and the problem of the large-scale structures formation.

3.3.2 Exponential laws

We choose an exponential law for the isotropic part of the metric and the potential:

$$\begin{aligned} e^\Omega &= e^{mt} \\ U &= U_0 e^{nt} \end{aligned} \tag{3.16}$$

U_0 is a constant. When the Universe isotropizes and undergoes expansion, we get a De-Sitter like behaviour for its dynamics and thus exponential inflation. This justifies the importance of this case which can also be considered as a limiting case of the polynomial inflation with $m \to +\infty$. Moreover, in FLRW models, the association of exponential laws for the scale factor and the potential is recovered in [43] where a superpotential is used and in [44] where conformal scalar field cosmologies are considered.

1. ***gravitational function***
   The gravitational function is written:

$$G^{-1} = C_1 e^{nt} + C_2 e^{-3mt} + C_3 e^{-2mt} \tag{3.17}$$

The C_i are integration constant. The Universe will isotropize at late times if $m > 0$, which means it is then expanding, or/and $n > -3m$. In this last case, asymptotically contracting Universe implies that the potential diverge. When the Universe isotropizes it tends toward a De-Sitter model. Consequently, when $m > 0$ and/or $n > -3m$, a De-Sitter Universe is a late times attractor for the class of theories defined by (3.16). This result can be compared, for the Bianchi type I model, to Wald results [49] which claims that, in the case of General Relativity with a scalar field and a cosmological constant, all the Bianchi models (except contracting Bianchi type IX) initially in expansion approach the isotropic De-Sitter solution.

The exact solution of the field equations can be found whatever m and n. We get for the anisotropic part of the metric:

$$\beta_\pm = \beta_{\pm 0}(mt - \ln[1 + C_3 e^{mt}(C_2 + C_1 e^{(3m+n)t})^{-1}])[m(C_2 + C_1 e^{(3m+n)t})]^{-1} + \beta_{\pm 1} \quad (3.18)$$

If we choose the early times at $t \to -\infty$, the functions $\beta_\pm$ tend toward linear law of the proper time or constant. This means that at early times the metric functions tend toward exponential laws of the proper time which can be compared to an anisotropic De Sitter Universe.

The only asymptotical behaviour for the metric functions which could be common between the case of this subsection and the previous one is an asymptotical static Universe. A necessary condition is then $m = 0$. Then, we see from (3.18) that the Universe can not isotropize and thus, asymptotically static Universe is not possible for the class of theories defined by (3.16). This last result excludes that General Relativity with a scalar field and a cosmological constant, defined by $m = 0$ and $n = 0$, isotropize at late time with a constant scale factor. This is in accordance with Wald results.

The late times behaviours of the classes of theories described in subsections 3.3.1 and 3.3.2 are represented on figure 3.1.

To our knowledge the results get in this last section are new and most of then can be extended to any functions e^Ω and U tending asymptotically toward the forms examined above. In [42], the same type of applications have been made in the Einstein frame. It was shown that the Universe tends toward an isotropic De-Sitter model when the potential tends toward a constant and reciprocally. In the present paper, we can see that such behaviour also arises when the potential diverges. In the same way, it was shown that the Universe isotropizes when its isotropic part tends toward a power law behaviour of the proper time if the scalar field is defined when the metric functions diverge but we had not get conditions on the exponent of the power law representing e^Ω. Moreover in subsection 3.3.1 we have also shown that Universe can isotropize toward a static model which is not possible in the Einstein frame. This underlines that HST is not dynamically equivalent to General Relativity with a scalar field and that new dynamical behaviours can be found.

3.4 Conclusion

In this work, we have determined, with help of quadrature, the solution of the field equations of the HST with a potential in the Bianchi type I model when we know the form of the potential and the isotropic part of the metric as some functions of the proper time. The first result we get is that the Universe isotropize when the integral of $Ge^{-3\Omega}$ tends asymptotically toward a constant. We had already obtained it in [42] by help of Hamiltonian formalism. Physically, it means that the 3-volume of the Universe have to grow faster than the gravitational function. It is in accordance with our present Universe which is expanding with a probably constant gravitational function. The Brans-Dicke coupling function can be evaluated as a function of the proper time and finally of the scalar field if it is an invertible function of t. However we have not study any particular form of ω since the dynamical properties of the Universe does not depend on it.

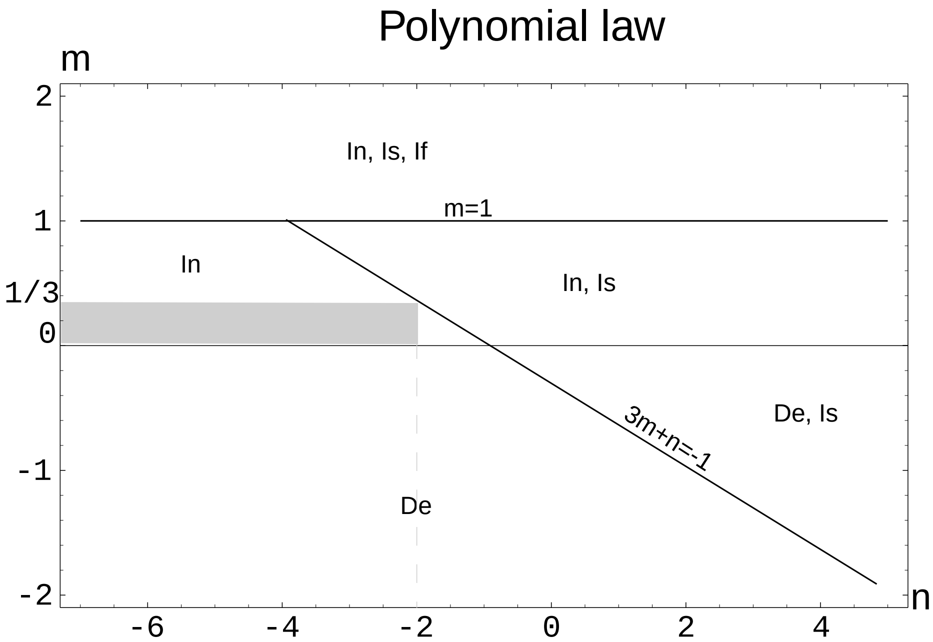
We have looked for the exact solutions of two classes of theories respectively defined by power and exponential laws of the proper time for e^Ω and U. They lead to isotropic Universe with power or exponential inflation and are linked, among others, to the presence of superpotential or conformal scalar field cosmologies. Of course, the forms we have chosen for e^Ω and U are particular ones. However most of the results we obtained should stay true for any theory whose the isotropic part of the metric and the potential asymptotically behave like these described in section 3.3. Particularly power laws of t are very interesting since from a mathematical point of view, any solutions which can be developed as power law series can be approximated in this way. Thus our results and the assumptions that the Universe be isotropic and undergoes inflation at late times could constraint any scalar tensor theories whose anisotropic part of the metric and potential can be developed as power series of the proper time or as exponential of t. Lets note also that from a physical point of view, power laws of the proper time are good approximations for the behaviour of the

3-volume at late times, when the solutions of the field equations approach FLRW ones, or at early times when the singularity is described by Kasnerian behaviour.

When the potential and the isotropic part of the metric are written as functions of power of t (respectively m and n), the gravitational function is the sum of three powers of the proper time. Then, the Universe isotropizes when $m > 1$ or $3m + n > -1$. In these two cases, an isotropic Universe with a power law for the metric functions represents the late times attractor. In the first case, the Universe will undergo inflation. In the second case, the presence of inflation at late times will imply $n < -4$. The opposite is not true.
We can not found the exact form of the anisotropic part of the metric for any values of m and n. However some interesting cases can be studied. The first one corresponds to an asymptotical coasting universe for which $m = 1$. Then, at late times the dynamical behaviour of the metric functions when universe isotropizes, i.e. for $n > -4$, is a linear law of t. In the FLRW case, this theory does not suffer of the age problem. A second case corresponds to a Universe with a cosmological constant ($n = 0$) which tends toward a power law of times with the same form as the solution for the flat radiation dominated Universe in the isotropic case ($m = 1/2$). For these two values of n and m, the Universe will always isotropize at late times. This theory is thus able to build a bridge between an anisotropic Bianchi type I Universe with a cosmological constant and an isotropic one behaving dynamically like a flat isotropic radiation dominated Universe. In a general way, when we consider a cosmological constant and a power law for the isotropic part of the metric , if $m > 0$, the Universe isotropize at late times and the gravitational function behaves like t^{-2} and vanishes. Moreover, if instead of a cosmological constant we choose a potential behaving like t^{-2}, the gravitational function will tend toward a constant instead of vanishing which is a physically interesting situation since it is what we observe for G. Remark that, whatever m and n, the only way to get asymptotically a non-vanishing gravitational constant with an expanding Universe is to choose $m = 1/3$ and $n \leq -2$ or $n = -2$ and $m \geq 1/3$. Hence, if we want to get at late times an inflationary Universe with a gravitation constant, we have to choose a potential behaving as t^{-2}. Note that the potential will then vanish at late times and will diverge at early times ($t = 0$), thus solving the cosmological constant problem. The last case we study is this of an asymptotical isotropic and static Universe ($m = 0$). It will be a late times attractor, i.e. the Universe will always tend toward an isotropic and static Universe, if $n > -1$. This type of theory could help to solve age and large scale structures formation problems.
When the potential and the isotropic part of the metric are written as functions of exponential of t (respectively m and n), G^{-1} is then the sum of three exponentials of t. The Universe will isotropize at late times if $m > 0$ or/and $n > -3m$. Under these conditions for m and n, a De-Sitter Universe is a late times attractor. In the first case, this always give birth to an expanding Universe. In the second case, if the potential asymptotically vanishes at late times as it could be the case for our present Universe, it can be contracting or expanding Universe.
It is possible to calculate the exact solutions of the metric functions, i.e. the anisotropic part of the metric, for any m and n. Thus, we have remarked that this class of models can not isotropize asymptotically toward a static Universe ($m = 0$). However, in the neighbourhood of the singularity that we choose in $t \to -\infty$, all the metric functions tend toward an exponential of the scalar field giving birth to the counterpart of a De Sitter model for the anisotropic case, i.e. the metric functions behave as exponentials of t with different exponents.

The two classes of theories we present in this paper have large regions of the parameters plane (m,n) for which the Universe is able to isotropize and be in expansion with a vanishing potential (In this case, for exponential laws of U and e^{Ω}, the gravitational function always diverge. This is different when we consider power laws.). Such types of theories could solve the cosmological constant problem since their potentials decrease naturally when time increases. These regions of the plane (m,n) are shown in the figure 3.1.

This work can be extended in several ways. First, since $G^{-1}(\Omega, \sum_n U_n) = g_1(\Omega) + \sum_n g_2(\Omega,U_n)$ and $\omega(\phi,\Omega, \sum_n U_n) = \omega_1(\phi,\Omega) + \sum_n \omega_2(\phi\ ,\Omega,U_n)$, we are able from the dynamical properties of simple classes of theories defined by $\Omega(t)$ and $U(t)$ to deduce dynamical properties of more evoluted classes of theories defined by sum of these functions. Secondly, other types of physically interesting potentials can be studied such as these which tend toward a constant in an oscillating way as for instance $U = \sin t/t + U_0$. Finally, It would be interesting to extend this work to take into account the presence of a perfect fluid as matter field for the Universe. It will be the subject of future works.



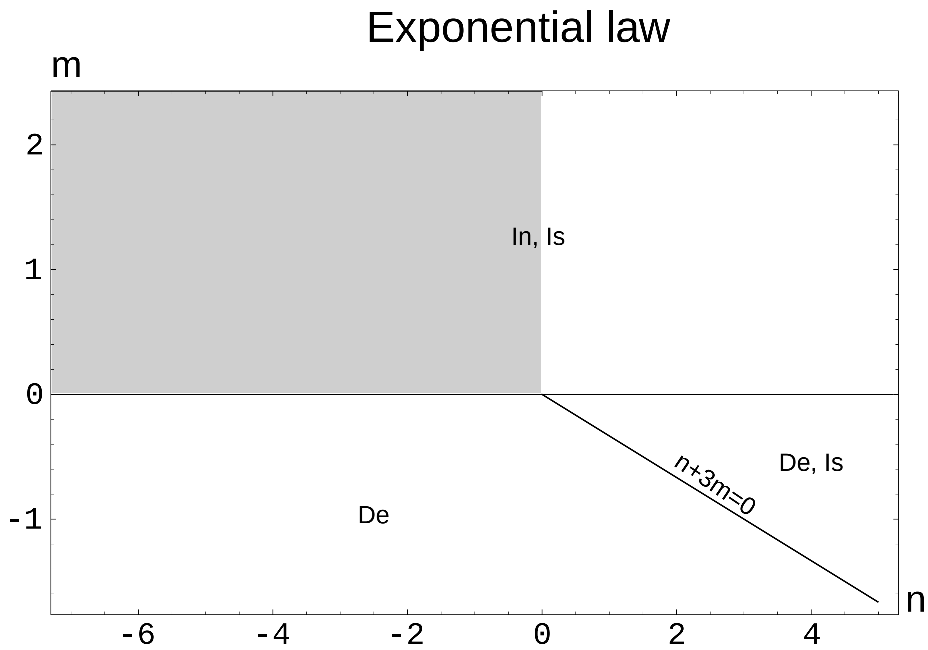


FIG. 3.1 – *These two figures summarise the asymptotical behaviours of the HST in the (m,n) plane when the potential and the isotropic parts of the metric are respectively power or exponential laws of the proper times. We have annoted 'De", 'In", 'Is"and 'If"the regions of the plane where the Universe is respectively decreasing, increasing, isotropic and inflationary at late times epochs. Gray regions represent the regions of the plane for which the gravitational function diverge at late times.*

Chapitre 4

Dynamique asymptotique du modèle de Bianchi de type I: formalisme Lagrangien 1(1 article)

Dans ce chapitre, nous considèrerons la théorie tenseur-scalaire définie par

$$L = \phi R - \frac{\omega(\phi)}{\phi}\phi_{,\mu}\phi^{,\mu}$$

G étant la fonction de gravitation et ω la fonction de Brans-Dicke, toutes deux dépendantes du champ scalaire ϕ. Notre but sera d'étudier le signe des dérivées premières et secondes des fonctions métriques en fonction de ω et de constantes d'intégration afin de comprendre qu'elles sont les théories compatibles avec un Univers en expansion et avec une accélération de sa dynamique. A l'époque où nous avons écrit cet article, la présente accélération de l'expansion de notre Univers n'avait pas encore été détectée, c'est la raison pour laquelle il n'en est pas fait mention dans l'article et pourquoi la notion d'accélération est systématiquement reliée à celle d'inflation.
Nous appliquerons nos résultats à trois théories tenseur-scalaires définies par:

$$\begin{aligned}
3 + 2\omega &= \phi_c^2\phi^{2m} \\
2\omega + 3 &= m \mid ln\phi/\phi_0 \mid^{-n} \\
2\omega + 3 &= m \mid 1 - (\phi/\phi_0)^n \mid^{-1}
\end{aligned}$$

Cette méthode apparaît plus efficace que la recherche de solutions exactes dans le sens où le comportement asymptotique de l'Univers (expansion, accélération, contraction, etc) peut être déterminé pour de larges classes de théories tenseur-scalaires. Cependant elle donne des résultats purement qualitatifs et ne nous permet pas encore d'obtenir des résultats quantitatifs tel que le comportement asymptotique des fonctions métriques exprimé à l'aide du temps propre t ou des contraintes d'origine physiques telles que celles issues de la nucléosynthèse[50] ou de l'isotropisation. De plus, nous n'avons pas réussi à l'étendre à d'autres modèles que celui de Bianchi de type I.

Dynamical study of the empty Bianchi type I model in generalised scalar-tensor theory.

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abstract:

A dynamical study of the generalised scalar-tensor theory in the empty Bianchi type I model is made. We use a method from which we derive the sign of the first and second derivatives of the metric functions and examine three different theories that can all tend towards relativistic behaviours at late time. We determine conditions so that the dynamic be in expansion and decelerated at late time.

Keys Words: Bianchi I; scalar-tensor theory; dynamical study.

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4.1 Introduction.

The scalar-tensor theories of gravitation allow to the gravitational constant to vary. Such a phenomenon happens in a large number of theories which try to unify gravitation with the other interaction forces. In the vacuum case, the most general form of the action of the scalar-tensor theories is written [31] :

$$S = \int \left[F(\phi)R - 1/2(\nabla\varphi)^2 - U(\varphi)\right]\sqrt{-g}d^4x \tag{4.1}$$

where φ is a scalar field, $U(\varphi)$ a potential. We get General Relativity with $F(\varphi) = cte$ and Brans-Dicke theory with $U = 0$, $F(\varphi) = \varphi^2/8\omega$ and $\omega = cte$. When $F(\varphi)$ is anatically invertible [51] this action can always be written with a Brans-Dicke scalar field. Putting $\phi = F(\varphi)$ and $\omega(\varphi) = F/\left[2(dF/d\varphi)^2\right]$), we get :

$$S = \int \left[\phi R - \frac{\omega(\phi)}{\phi}(\nabla\phi)^2 - U(\phi)\right]\sqrt{-g}d^4x \tag{4.2}$$

We will take $U(\phi) = 0$ so that we can obtain a Newtonian limit for the weak fields [29]. Techniques to find exact or asymptotic solutions to the field equations derived from action (4.2), with or without matter, in an anisotropic Universe, by means of a conformal transformation, have been described in [29]. Exact solutions and asymptotic behaviours of the scale factor have been analysed for the generalised scalar-tensor theory in FLRW model with matter in [52]. Dynamical studies have been made for Brans-Dicke theory in a FLRW model in [53, 54, 55]. Here, we will work in an empty Bianchi type I Universe. We will introduce new variables, write the field equations with their first derivatives and then perform an analysis to get analytically the sign of the first and second derivatives of the metric functions, without asymptotic methods, whatever $\omega(\phi)$. Hence we will get the qualitative form of these functions in the Brans-Dicke frame for any time: are they increasing or decreasing, do extrema exist and if so, how many, is there inflation, do they tend towards a power law type, etc.

In section 4.2, we write the field equations of the vacuum Bianchi type I model with the new variables. In section 4.3, we study particular values of these variables and in section 4.4 we describe the method which gives the sign of the first derivatives of the metric functions, depending on the form of $\omega(\phi)$. In section 4.5, we apply our method to three different forms of the coupling function which are all such that $\omega \to \infty$ and $\omega_\phi\omega^{-3} \to 0$ if we adjust some of their parameters. These two limits ensures that the PPN parameters converge towards values in agreement with the observational data [56, 57]. Thus the different theories, corresponding to different choices of the coupling $\omega(\phi)$, converge towards relativistic behaviours. In section 4.6, we examine the three metric functions and under what conditions they are increasing or decreasing together, etc. In the section (4.7), we describe the method giving the sign of the second derivatives of the metric functions and examine in which conditions they can be decelerated at late time. We apply our results to the coupling functions of section 4.5.

4.2 The field equations

The metric is:

$$ds^2 = -dt^2 + a^2(\omega^1)^2 + b^2(\omega^2)^2 + c^2(\omega^3)^2 \tag{4.3}$$

where the ω^i are the 1-forms of the Bianchi type I model, t the proper time and $a(t)$, $b(t)$, $c(t)$ the metric functions depending on t. We define the τ time as:

$$d\tau = abcdt$$

and then, the field equations and the Klein-Gordon equation are written:

$$\begin{aligned}
\frac{a''}{a} - \frac{a'^2}{a^2} + \frac{a'}{a}\frac{\phi'}{\phi} - \frac{1}{2}\frac{\omega'}{3+2\omega}\frac{\phi'}{\phi} &= 0 \\
\frac{b''}{b} - \frac{b'^2}{b^2} + \frac{b'}{b}\frac{\phi'}{\phi} - \frac{1}{2}\frac{\omega'}{3+2\omega}\frac{\phi'}{\phi} &= 0 \\
\frac{c''}{c} - \frac{c'^2}{c^2} + \frac{c'}{c}\frac{\phi'}{\phi} - \frac{1}{2}\frac{\omega'}{3+2\omega}\frac{\phi'}{\phi} &= 0
\end{aligned} \tag{4.4}$$

$$\frac{a'}{a}\frac{b'}{b} + \frac{a'}{a}\frac{c'}{c} + \frac{b'}{b}\frac{c'}{c} + \frac{\phi'}{\phi}(\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c}) - \frac{\omega}{2}(\frac{\phi'}{\phi})^2 = 0 \tag{4.5}$$

$$\phi'' = -\frac{\omega'\phi'}{3+2\omega} \tag{4.6}$$

We integrate (4.6) and get :

$$A\phi'\sqrt{3+2\omega} = 1 \tag{4.7}$$

A being an integration constant. We see in this last expression that the coupling function must be superior to -3/2 so that the square root is real. We use (4.6) to introduce the second derivative of the scalar field in (4.4) and put:

$$\alpha = \frac{a'}{a}\phi,\ \beta = \frac{b'}{b}\phi,\ \gamma = \frac{c'}{c}\phi,\ \phi' = \Phi \tag{4.8}$$

After integrating, the field equations become:

$$\begin{aligned}
\alpha + \frac{1}{2}\Phi &= \alpha_0 \\
\beta + \frac{1}{2}\Phi &= \beta_0 \\
\gamma + \frac{1}{2}\Phi &= \gamma_0 \\
\alpha\beta + \alpha\gamma + \beta\gamma + \Phi(\alpha+\beta+\gamma) - \frac{1}{4}(A^{-2} - 3\Phi^2) &= 0
\end{aligned} \tag{4.9}$$

α_0, β_0, γ_0 being integration constants. The constraint imposes the condition :

$$\alpha_0\beta_0 + \alpha_0\gamma_0 + \beta_0\gamma_0 = (4A^2)^{-1} \tag{4.10}$$

The physical solutions are such that the metric functions and the scalar field are positive. Hence, the sign of the variables α, β, γ will be the same as the sign of the first derivative of the metric functions. The sign of Φ will be the same as ϕ'. Negative scalar fields have already been considered in [58] but it means that, in the Einstein frame, the gravitational constant will be negative. For this reason, many authors deal with positive scalar fields. We will do the same, but the method can easily be extended to negative ones. In what it follows, we will consider only the metric function a. What we write for a will be valid for b and c. Let us say a few words about exact solutions [29] of the field equations. From (4.9), we can easily show that:

$$a = exp(\int \frac{\alpha_0}{\phi}d\tau + cte)\phi^{-1/2} \tag{4.11}$$

The scalar field can be calculated by integrating and inverting (4.12):

$$d\tau = A\int \sqrt{3+2\omega}d\phi \tag{4.12}$$

Therefore, we can obtain exact solutions of the metric functions for the simple form of the coupling function.

What is the link between the results we will obtain in the τ time and the behaviours of the metric functions in the t time. Since $a(\tau) = a(\tau(t)) = a(t)$, the amplitudes of the metric functions will be the same in both τ and t times. Moreover as:

$$da/dt = da/d\tau d\tau/dt = da/d\tau(abc)^{-1} \tag{4.13}$$

with $abc > 0$, the sign of the first derivatives of the metric functions will not be different in τ or t time. Of course the amplitudes of all the derivatives will be different. While it will always be possible to determine asymptotically the amplitudes of a', this will not be the same for $\dot{a}$.

Therefore, as we are mainly interested by the sign of a', a'' and $\ddot{a}$, this is not important. The sign of the second derivatives will be different in both times since

$$d^2a/dt^2 = \ddot{a} = [a'' - a'(a'/a + b'/b + c'/c)]\,(abc)^{-2} \tag{4.14}$$

an overdot denoting differentiation with respect to t.

For these reasons, all that we will say about the sign of the first derivatives will apply to both t and τ time. Hence, results of section 4.3, 4.4, 4.5, 4.6 and in particular table 4.1 (except the sign of the second derivative of the scalar field which will be different by ϕ'' in the t time) will not change in t time since they depends on the sign of constants or first derivative of ω with respect to ϕ. In section 4.7, where we will deal with the sign of the second derivatives, we will study separately the sign of a'' and $\ddot{a}$.

Another difference between τ and t time is that, for instance, t can diverge for a finite value of τ. It can, for instance, transform a Universe that exists during a finite τ time into a Universe which would exist in an infinite t time. But we will not pay attention to this type of phenomenon in our study. In fact, in most cases, we will use ϕ as a time coordinate, particularly in section 4.5 and 4.7, and so we will have no need to know the intervals of τ or t.

4.3 Study of the first derivative of a metric function.

We consider the first equation of (4.9). The solution of this equation in the (α,Φ) plane is represented by a straight line. We have two cases depending on the sign of α_0, which are represented on graph 1. To describe the variations of the metric function a, we have to study the dynamic of a point (α,Φ) on this straight line so that we know the sign of α and hence, this of a' during the time evolution. The straight line cuts the Φ axe at $(\alpha,\Phi) = (0,2\alpha_0)$ and the α axe at $(\alpha,\Phi) = (\alpha_0,0)$. In $(0,2\alpha_0)$, we have $\alpha = 0$. This means that :

- the metric function a reaches an extrema if the motion of the point (α,Φ) on the straight line is such that the sign of α change. It is an inflexion point for the metric function, if the motion of the point (α,Φ) on the straight line changes direction when it reaches $(0,2\alpha_0)$.

- If the motion of the (α,Φ) point on the straight line is such that it tend asymptotically towards $(0,2\alpha_0)$ then a possible explanation is that the scalar field vanishes or that $a \propto \tau$.

In $(\alpha_0,0)$, the first derivative of the scalar field disappears. We will show below that the scalar field is a monotone function of τ. Hence , $\phi' = 0$ can be an inflexion point for ϕ in the τ time if the motion of the point (α,Φ) changes direction after reaching $(\alpha_0,0)$. Otherwise it means that the scalar field tends towards a constant. In this last case, we have $\phi \rightarrow \phi^* = cte$ and (4.7) shows that $\omega \rightarrow \infty$. If we put $\phi = \phi^*$ in the field equations (4.9), the metric functions are written $a = e^{\alpha_0\phi^{*-1}(\tau-\tau_0)}$, $b = e^{\beta_0\phi^{*-1}(\tau-\tau_0)}$, $c = e^{\gamma_0\phi^{*-1}(\tau-\tau_0)}$, and become in the proper time $a = a_0t^{p_1}$, $b = b_0t^{p_2}$, $c = c_0t^{p_3}$ with $\sum p_i = 1$ and $\sum p_i^2 = 1 - 2^{-1}A^{-2}(\alpha_0 + \beta_0 + \gamma_0)^{-2}$. Hence, when $(\alpha,\Phi) \rightarrow (\alpha_0,0)$, the metric functions tend towards a Kasnerian behaviour.

We can make the following general observations valid in the τ time: when $\Phi \notin [2\alpha_0,0]$, the more increasing (decreasing) the scalar field is, the more decreasing (increasing) the metric function will be. When $\Phi \in [2\alpha_0,0]$, the scalar field and the metric function increase (decrease) if $\alpha_0 > 0$ ($\alpha_0 < 0$).

The last remark will concern the representation, in the (α,Φ) plane, of the solutions of the first equation in (7). If we take as a convention that $\sqrt{3+2\omega} > 0$, equation (4.7) shows that the sign of $\phi' = \Phi$ depends on the sign of the integration constant A. Hence the solution represented in figure 1 by the straight line is physically composed of two separate solutions represented by two half-line, one corresponding to $A > 0$ and then $\Phi > 0$ and the other to $A < 0$ and then $\Phi < 0$. So, to the first equation of (4.9) correspond four types of behaviours for the metric function and the scalar field, depending on the sign of α_0 and A. We will see below that each of them can be split again in two cases depending on the sign of $\Phi' = \phi''$. These four solutions are illustrated in figure 2. In this figure, {1}, {2}, {3}, {4} correspond to the four half-lines which represent the four physically different solutions of the first equation of (4.9). (τ_1) and (τ_2) represent the finite or infinite values of the time τ for which (α,Φ) is equal to $(0,2\alpha_0)$ and $(\alpha_0,0)$. In what it follows, we will consider the motion of a point (α,Φ) on each of the four half-lines. It depends on the form of the coupling function $\omega(\phi)$. To determine it, we need an equation to know how and under which conditions Φ varies.

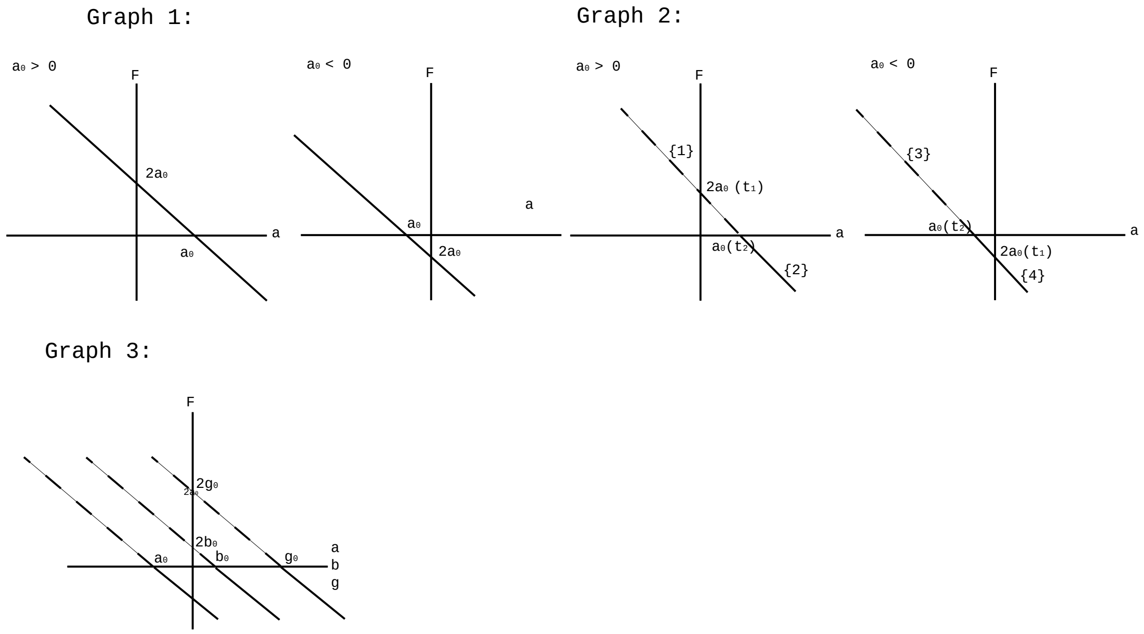

FIG. 4.1 – *.Figure 1 : solution of the first equation of (4.9) in the α,Φ plane depending on the sign of α_0.*

Figure 2 : the four different physically solutions of the first equation of (4.9).

Figure 3 : representation of all the solutions of the equations (4.9) in the $((\alpha,\beta,\gamma),\Phi)$ plane.

4.4 Study of the metric functions and scalar field variations depending on the form of $\omega(\phi)$

We have $d\tau = abcdt$ with $abc > 0$. Hence τ is an increasing function of t and the variations of the metric functions in the τ time will be the same in the t time. From (4.6), we deduce the equation which gives the variation of Φ depending on $\omega(\phi)$:

$$\Phi' = -\frac{\omega_\phi(\phi')^2}{3+2\omega} \tag{4.15}$$

with $\omega_\phi = \omega'/\phi' = d\omega/d\phi$. $3+2\omega$ is positive since $\omega > -3/2$. Then, the sign of Φ' depends on the sign of ω_ϕ which is independent of the time we consider, namely τ or t (of course $\Phi' = \phi''$ and the sign of $\ddot{\phi}$ will be different in the t time. But this is not important here since our final aim is to determine the sign of the first derivatives of the metric functions which does not change in t time). So the results we will find and which depend on the sign of the variations of Φ will be valid in both t and τ times. Hence, if ω_ϕ has a constant sign, the motion of the point (α,Φ) on each half-line will be monotone otherwise its direction will change depending on the sign of ω_ϕ. We now study the case where $\omega(\phi)$ is a monotone function and get eight different behaviours for the scalar field and the metric function corresponding to the split of each of the 4 previous cases in two cases. First, we consider that the coupling function is an increasing function of the scalar field. Then, $\omega_\phi > 0$ and from (4.15) we deduce that $\Phi' = \phi'' < 0$. Consequently, the motion of the point (α,Φ) on the half-lines will be such that Φ decrease. Then, if we are on the half-line $\{1\}$, the point (α,Φ) moves from the left to the right. In the same time, τ increases and then we deduce that $\tau_1 < \tau_2$. On $\{1\}$ we have $\Phi = \phi' > 0$: the scalar field is an increasing function of τ. When $\Phi \to +\infty$, $\alpha < 0$. α remains negative until $(\alpha,\Phi) = (0,2\alpha_0)$, which means $\tau = \tau_1$, and when $\Phi \in [0,2\alpha_0]$, α becomes positive. So, we deduce that the metric function is first decreasing until $\tau = \tau_1$ and then increases when $\tau > \tau_1$ until $\tau = \tau_2$, the value of τ for which the scalar field becomes a constant: the metric function can have a minimum (but it

Sign of $(\omega_\phi, A, \alpha_0)$	Variation of ϕ	Variation of $\alpha(\tau)$	half-line number	direction of the monotone motion of the (α, Φ) point	type of behaviour number
(+,+,+)	$\phi' > 0, \phi'' < 0$	minimum in τ_1 when $\Phi = 2\alpha_0$	$\{1\}$	left to right	$\{1\}$
(+,-,+)	$\phi' < 0, \phi'' < 0$	increasing	$\{2\}$	left to right	$\{2\}$
(+,+,-)	$\phi' > 0, \phi'' < 0$	decreasing	$\{3\}$	left to right	$\{3\}$
(+,-,-)	$\phi' < 0, \phi'' < 0$	minimum in τ_1 when $\Phi = 2\alpha_0$	$\{4\}$	left to right	$\{4\}$
(-,+,+)	$\phi' > 0, \phi'' > 0$	maximum in τ_1 when $\Phi = 2\alpha_0$	$\{1\}$	right to left	$\{1'\}$
(-,-,+)	$\phi' < 0, \phi'' > 0$	increasing	$\{2\}$	right to left	$\{2'\}$
(-,+,-)	$\phi' > 0, \phi'' > 0$	decreasing	$\{3\}$	right to left	$\{3'\}$
(-,-,-)	$\phi' < 0, \phi'' > 0$	maximum in τ_1 when $\Phi = 2\alpha_0$	$\{4\}$	right to left	$\{4'\}$

TAB. 4.1 – *The eight types of behaviours of the scalar field and metric function when the coupling constant is a monotone function of the scalar field. Note that the sign of the second derivative of ϕ with respect to τ or t will not be the same. But the signs of all the first derivatives will stay the same.*

is not necessarily true as we will see below). The same type of reasoning can be applied when we consider the half-lines $\{2\}$, $\{3\}$ and $\{4\}$.

If now we consider that the coupling function is a decreasing function of the scalar field, we have $\omega_\phi < 0$ and $\Phi' = \phi'' > 0$. The point (α,Φ) moves from the right to the left on each of the four half-lines and we have $\tau_2 < \tau_1$. The same reasoning as in the case $\omega_\phi > 0$ will hold. Hence we get four more cases. Table 4.1 summarises these eight cases : we give the sign of the triplet (ω_ϕ,A,α_0), independent of the time we consider (t or τ), the scalar field and metric function variations, the direction of the motion of the point on each half-line and we allocate a number for each behaviour. Another condition has to be fulfilled in the cases $\{1\}$, $\{1'\}$, $\{4\}$, $\{4'\}$, to have necessarily an extremum: we have to check if the value $\Phi = 2\alpha_0$ belongs to the interval in which Φ varies. For this purpose, we rewrite the equation (4.7) :

$$A\Phi\sqrt{3+2\omega} = 1 \tag{4.16}$$

We determine the interval in which the scalar field ϕ varies by imposing the conditions $\sqrt{3+2\omega} > 0$ and $\phi > 0$. Then from (4.16) we deduce the interval for Φ. The condition for an extremum to exist for the behaviours of type $\{1\}$, $\{1'\}$, $\{4\}$ and $\{4'\}$ will be that this last interval contains the value $2\alpha_0$. One can also check if the value of the scalar field corresponding to $3 + 2\omega = (2\alpha_0 A)^{-2}$ beholds to the interval in which ϕ varies.

Now, we consider the case where the coupling function $\omega(\phi)$ is not a monotone function of the scalar field. It means that the sign of ω_ϕ will change during the evolution of the dynamic. In the interval of time where ω_ϕ will be positive, we will have behaviours of type $\{1\}$, $\{2\}$, $\{3\}$ or $\{4\}$ and when it becomes negative the metric function and the scalar field will behave respectively as $\{1'\}$, $\{2'\}$, $\{3'\}$ or $\{4'\}$. Hence, the behaviours of the metric function when the coupling function is not monotone will be a succession of behaviours of type $\{i\}$+$\{i'\}$+$\{i\}$+$\{i'\}$..., the repetitions of the scheme $\{i\}$+$\{i'\}$ depending on the number of zero of ω_ϕ.

Note that to achieve our goal, that is determine the variation (sign of the first derivative) of the metric function, we used quantities such that the second derivative of the scalar field or the amplitude of its first derivative are not invariant when we change time coordinate from τ to t. But these two quantities can always be written as function of ω_ϕ or ω which are independent of time coordinate. Therefore our method is in agreement with the fact that the sign of the first derivative of the metric function is the same in τ or t time.

In the next section we will consider several forms of the coupling function with a decreasing scalar field, i.e. $A < 0$.

4.5 Applications.

We are going to examine the variations of the metric functions with three different forms of the coupling function. The couplings we will consider are interesting for the following reasons. The first coupling is $3+2\omega=\phi_c^2\phi^{2m}$. When $m>0$ and $\phi\rightarrow\infty$ or $m<0$ and $\phi\rightarrow 0$, $\omega\rightarrow\phi^{2m}\rightarrow\infty$. When $m<-1/4$ and $\phi\rightarrow 0$ or when $m>-1/4$ and $\phi\rightarrow\infty$, $\omega_\phi\omega^{-3}\rightarrow 0$. Hence, asymptotically, the theory tends towards relativistic behaviours at late time ($\phi\rightarrow 0$) when $m<-1/4$. When the scalar field becomes infinite, $\omega(\phi)$ tends towards a power law that corresponds to a power or exponential law for $F(\varphi)$ (see (4.1)). Power Law for $\omega(\phi)$ have been studied in [59]. This class of theories is also in agreement with the constraints imposed by the slow logarithmic decrease of the gravitational constant $(dG/dt)G^{-1}$. The two other laws, $2\omega+3=m\mid\ln(\phi/\phi_0)\mid^{-n}$ and $2\omega+3=m\mid 1-(\phi/\phi_0)^n\mid^{-1}$ have been studied in [52] in a FLRW Universe. For the first one, we recover the values of the PPN parameters in General Relativity when $\phi\rightarrow\phi_0$ if $n>1/2$, whereas for the second one there is no restriction on the value of the exponent n.

4.5.1 The theory $3+2\omega=\phi_c^2\phi^{2m}$

We have :

$$\omega_\phi=\phi_c^2 m\phi^{2m-1} \tag{4.17}$$

The expression $3+2\omega$ is positive for all positive values of the scalar field. Hence ϕ varies in $[0,+\infty[$. From (4.16) we deduce that Φ varies in $]-\infty,0]$. If m is positive, $\omega_\phi>0$ and the metric function behaves as $\{2\}$ and $\{4\}$ whereas if m is negative, $\omega_\phi<0$, and it behaves as $\{2'\}$ and $\{4'\}$. In the Cases $\{2\}$ and $\{2'\}$, the metric function increases. In the case $\{4\}$ and $\{4'\}$, from (4.16) we deduce that the metric function has an extremum when the scalar field is equal to $(2\alpha_0 A\phi_c)^{1/m}$. This last value is always positive and then belongs to the interval in which the scalar field varies. We conclude that for the types $\{4\}$ or $\{4'\}$, the metric function will always have respectively a minimum or a maximum.

4.5.2 The theory $2\omega+3=m\mid ln\phi/\phi_0\mid^{-n}$.

We restrict the parameters to $n>0$, $m>0$ so that $2\omega+3$ is positive. We will first consider the case where $\phi>\phi_0$. Then, we can write:

$$2\omega+3=m(ln\phi/\phi_0)^{-n} \tag{4.18}$$

ω_ϕ is always negative and $\Phi\in\,]-\infty,0]$. Hence, if $\alpha_0>0$, the metric function is increasing. If $\alpha_0<0$, the metric function will always have a maximum since $\Phi=2\alpha_0$ belongs to the interval where Φ varies.

If we chose for ϕ the interval $[0,\phi_0]$, the metric function has a minimum if $\alpha_0<(2A\sqrt{m})^{-1}$. Otherwise, it is increasing.

4.5.3 The theory $2\omega+3=m\mid 1-(\phi/\phi_0)^n\mid^{-1}$.

We restrict the parameters to $n>0$, $m>0$ and will take first $\phi>\phi_0$. Hence we have:

$$2\omega+3=m\left[(\phi/\phi_0)^n-1\right]^{-1} \tag{4.19}$$

ω_ϕ is always negative. If the integration constant α_0 is positive, the metric function is increasing, whereas if α_0 is negative, since $\Phi\in\,]-\infty,0]$, the metric function will always have a maximum. If we choose $\phi\in[0,\phi_0]$, the metric function is still increasing when $\alpha_0>0$ but have a minimum if $\alpha_0<0$.

4.6 Behaviour of the three metric functions.

The graph 3 represents the solutions of the system equations (4.9) on the plane $((\alpha,\beta,\gamma),\Phi)$. We choose without loss of generality $\alpha_0<\beta_0<\gamma_0$. We distinguish four cases :

1. If $\Phi>2\gamma_0$, all the metric functions are decreasing.
2. If $\Phi\in[2\gamma_0,2\beta_0]$, the metric function associated with the largest of the integration constants is increasing whereas the two others are still decreasing.
3. If $\Phi\in[2\beta_0,2\alpha_0]$, the metric function associated with the smallest of the integration constants is the only one to be decreasing.
4. If $\Phi<2\alpha_0$, the three metric functions are increasing.

If i constants among α_0, β_0 and γ_0 are positive, we deduce from figure 3 that when ϕ is increasing, whatever the form of $\omega(\phi)$, only the i+1 first cases can exist, when ϕ is decreasing, whatever the form of $\omega(\phi)$, only the i+1 last cases can exist. Hence, in the case where α_0, β_0, γ_0 are positive constant and A is a negative one, all the metric functions will be increasing whatever the form of $\omega(\phi)$. But, if α_0, β_0, γ_0 are negative and A positive, all the metric functions will be decreasing. We deduce also that to get three increasing metric functions which tend towards a power law, that is $((\alpha,\beta,\gamma),\Phi) \rightarrow ((\alpha_0,\beta_0,\gamma_0),0)$, when τ(and thus t) increases, a necessary condition will be that α_0, β_0, γ_0 be positive , A and ω_ϕ have the same sign.

4.7 Study of the second-derivative of the metric function

In the FLRW models, a positive sign of the first and second derivatives of the scale factor with respect to the cosmic time is the sign of inflation: the expansion in the t time is accelerated. Inflation in generalised scalar-tensor theory and in FLRW models has been studied in [27] and [28]. It seems to be noteworthy that it happens without a cosmological constant or potential. One can talk about inflation only when the second derivatives of the metric functions with respect to t are positives. First, we are going to describe a method giving the sign of the second derivative of the metric function with respect to τ from the knowledge of ω and ω_ϕ. Hence, we will be able to completely determine the qualitative form of the metric function in the τ time. Second, we apply it and finally we will study the sign of $\ddot{a}$ and obtain conditions to have inflation in Bianchi type I model.

4.7.1 Study of a''

The first spatial component of the field equations is written :

$$\frac{a''}{a} = \frac{a'^2}{a^2} - \frac{a'}{a}\frac{\phi'}{\phi} + \frac{1}{2}\frac{\omega'}{3+2\omega}\frac{\phi'}{\phi} \tag{4.20}$$

$$\phi^2\frac{a''}{a} = \alpha^2 - \alpha\phi' + \frac{1}{2}\frac{\omega_\phi}{3+2\omega}\phi'^2\phi \tag{4.21}$$

But $\phi' = 1/(A\sqrt{3+2\omega})$, so we get :

$$\phi^2\frac{a''}{a} = \alpha^2 - \frac{\alpha}{A\sqrt{3+2\omega}} + \frac{1}{2}\frac{\omega_\phi}{(3+2\omega)^2}\frac{\phi}{A^2} \tag{4.22}$$

The sign of the left hand side of (4.22) is the same as a''. The right hand side of equation (4.22) is an equation of degree two in α. Hence, we have to know the sign of this equation in order to obtain the sign of a'', i.e. to determine its roots. It is important to recall that α can be expressed as a function of the scalar field. We get :

$$\alpha = \alpha_0 - \frac{1}{2}\phi' = \alpha_0 - \frac{1}{2}\frac{1}{A\sqrt{3+2\omega}} \tag{4.23}$$

Now we calculate the determinant of the second degree equation (4.22) :

$$\Delta = \frac{1}{A^2(3+2\omega)} - 2\frac{\omega_\phi}{(3+2\omega)^2}\frac{\phi}{A^2} \tag{4.24}$$

If Δ is negative, the second degree equation is positive for all value of α and a'' is positive. Then the dynamic of the metric function is accelerated (this is not inflation since the sign of a'' and $\ddot{a}$ are not necessarily the same). If Δ is positive, the second degree equation has two real roots α_1 and α_2. From (4.24), we deduce that $\Delta < 0$ if :

$$\omega_\phi > \frac{3+2\omega}{2\phi} \tag{4.25}$$

The condition (4.25) will be true for the three metric functions. It does not depend on a specific parameter of one of these functions. Hence, when (4.25) is true, the dynamic of the three metric functions in the τ time is accelerated. If now we consider $\Delta > 0$, we find two roots :

$$\alpha_{1,2} = \left(\frac{1}{A\sqrt{3+2\omega}} \pm \sqrt{\frac{1}{A^2(3+2\omega)} - \frac{2\omega_\phi}{(3+2\omega)^2}\frac{\phi}{A^2}}\right)/2 \tag{4.26}$$

With the form of the coupling function, one can deduce the conditions so that a'' be positive or negative. By conditions we mean the values of the scalar field and of the different parameters defining the form of the coupling function, which rule the sign of a''. To get this sign, we have to know the sign of:

$$\alpha_{1,2}(\phi) - \alpha(\phi) = -\alpha_0 + (2A\sqrt{3+2\omega})^{-1}\left[2 \pm \sqrt{1 - 2\omega_\phi \phi(3+2\omega)^{-1}}\right] \tag{4.27}$$

When $\alpha_1 - \alpha$ and $\alpha_2 - \alpha$ have the same sign, equation (4.22) is positive and thus a'' is positive; otherwise, it means that $\alpha \in [\alpha_2,\alpha_1]$ and then a'' is negative. At late time, if ϕ_{RG} is the value of the scalar field for which $\omega \to \infty$ and $\omega_\phi \omega^{-3} \to 0$ (which ensures the theory is compatible with the observation) we deduce from (4.27) that a necessary and sufficient condition for the dynamic of the metric function to be decelerated in the τ time, will be:

$$\lim_{\phi \to \phi_{RG}} \omega_\phi < -2\alpha_0^2 A^2 (3+2\omega)^2 \phi^{-1} \tag{4.28}$$

4.7.2 Applications.

Theory $3 + 2\omega = \phi_c^2 \phi^{2m}$

Remember that for this form of $3+2\omega$ we have $\phi \in [0, +\infty[$. We continue to choose $A < 0$ in order to have a decreasing scalar field. We get :

$$\alpha = \alpha_0 - \frac{1}{2}\frac{\phi^{-m}}{A\phi_c} \tag{4.29}$$

$$\alpha_{1,2} = \frac{\phi^{-m}(1 \pm \sqrt{1-2m})}{2A\phi_c} \tag{4.30}$$

The condition (4.25) is satisfied when $m > 1/2$: in this case we always have a'', b'' and c'' positive. When $m < 1/2$, we have to determine the sign of :

$$\alpha_{1,2} - \alpha = \frac{\phi^{-m}(2 \pm \sqrt{1-2m})}{2A\phi_c} - \alpha_0 \tag{4.31}$$

We will always have $\alpha_1 < \alpha_2$.

- If $\alpha_0 = 0$, we have $\alpha > \alpha_1$ for all values of the scalar field. If $m < -3/2$, from equation (4.31) we deduce that $\alpha_2 < \alpha < \alpha_1$ and thus $\alpha'' < 0$. If $m \in [-3/2,1/2]$, we get $\alpha > \alpha_{1,2}$ and then $a'' > 0$. Now we consider general case where $\alpha_0 \neq 0$.

– If $m < 0$,
  – if $\alpha_0 > 0$, when $\phi \to \infty$, $\alpha > \alpha_1$. If $m \in [-3/2,0]$, $\alpha > \alpha_{1,2}$ and if $m < -3/2$, $\alpha \in [\alpha_1,\alpha_2]$. Then the scalar field decreases and when $\phi \to 0$, $\alpha > \alpha_{1,2}$.
  – If $\alpha_0 < 0$, when $\phi \to \infty$, if $m \in [-3/2,0]$, $\alpha > \alpha_{1,2}$, if $m < -3/2$, $\alpha \in [\alpha_1,\alpha_2]$. When the scalar field decreases and $\phi \to 0$, $\alpha < \alpha_{1,2}$.

  Hence, we deduce that :
  – If $\alpha_0 > 0$,
    – if $m \in [-3/2,0]$, we have $a'' > 0$,
    – if $m < -3/2$, we have first $a'' < 0$ and then $a'' > 0$.
  – If $\alpha_0 < 0$,
    – if $m \in [-3/2,0]$, we have $a'' > 0$, then $a'' < 0$ and finally $a'' > 0$,
    – if $m < -3/2$, we have $a'' < 0$ and $a'' > 0$.

– If $m \in [0,1/2]$,
  We will always have $\phi^{-m}(2 - 1\sqrt{1-2m}) > 0$. When $\phi \to \infty$, α is larger than $\alpha_{1,2}$ if $\alpha_0 > 0$ or smaller if $\alpha_0 < 0$. For all value of α_0, when ϕ decreases and tends towards 0, we have $\alpha > \alpha_{1,2}$.
  Hence, we deduce that if $\alpha_0 < 0$, first we have $a'' > 0$, then $a'' < 0$ and at last $a'' > 0$. If $\alpha_0 > 0$, we always have $a'' > 0$.

From the knowledge of a' (see 4.5.2) and a'' it is now easy to know qualitatively the behaviours of the metric function a, depending on its different parameters α_0 and m. We deduce from our qualitative analysis that:

– When $m \in [0,1/2]$ and $\alpha_0 > 0$, the metric function is increasing and accelerated. When $\alpha_0 < 0$, the metric function has a minimum. The branch before the minimum is accelerated whereas the branch after the minimum has an inflexion point and is accelerated in late time.

– When $\underline{m > 1/2}$, the dynamic of the metric function is always accelerated.
– When $\underline{m < 0}$ and $\alpha_0 > 0$, the metric function increases. It is accelerated if $m \in [-3/2,0]$. If $m < -3/2$, it is first decelerated and then accelerated: the metric function has an inflexion point. If $\alpha_0 < 0$, the metric function has a maximum. If $m \in [-3/2,0]$, the dynamic is accelerated in both late and early times whereas if $m < -3/2$, it is decelerated in early time and accelerated in late time.

Note that one can always obtain the value of the scalar field for which the sign of a'' changes by writing $\alpha_{1,2} - \alpha = 0$. We see that the theory $3+2\omega = \phi_c^2\phi^{2m}$ is always accelerated in late time in accordance with the relation (4.28).

The theory $2\omega + 3 = m \mid \ln\phi/\phi_0 \mid^{-n}$**.**

Here, we consider only the interval $[\phi_0,\infty[$ for the scalar field, ω_ϕ is always negative and then Δ is always positive. We have:

$$\alpha_{1,2} - \alpha = -\alpha_0 + (2A\sqrt{m})^{-1}(\ln\phi/\phi_0)^{n/2}(2 \pm \sqrt{1 + n\phi_0 \ln(\phi/\phi_0)^{-1}}) \tag{4.32}$$

When $\underline{\alpha_0 > 0}$, in early time, $\phi \to \infty$ and $\alpha > \alpha_{1,2}$. Then, at late time, when $\phi \to \phi_0$, if $n > 1$, we have again $\alpha > \alpha_{1,2}$ and then the metric function increases and is accelerated whereas if $n \in [0,1]$, we have $\alpha \in [\alpha_1,\alpha_2]$. Then, the metric function increases but have an inflexion point. It is decelerated at late time. When $\underline{\alpha_0 < 0}$, the metric function has a maximum. If $n > 1$, the dynamic is both accelerated in early and late time whereas if $n \in [0,1]$, it is just accelerated in early time.

The theory $2\omega + 3 = m \mid 1 - (\phi/\phi_0)^n \mid^{-1}$**.**

Here again we consider the same interval for ϕ and Δ will be always positive. We have:

$$\alpha_{1,2} - \alpha = -\alpha_0 + (2A\sqrt{m})^{-1}\sqrt{(\phi/\phi_0)^{-n} - 1}(2 \pm \sqrt{1 + n(\phi/\phi_0)^n / \left[(\phi/\phi_0)^n - 1\right]}) \tag{4.33}$$

We get two important values for n: $n = 3$ or $n = 4A^2\alpha_0^2 m$.

– When $\alpha_0 > 0$, the metric function is increasing and its behaviour is accelerated if $n < (3,4A^2\alpha_0^2 m)$ or decelerated if $n > (3,4A^2\alpha_0^2 m)$. If the value of n is between $n = 3$ and $n = 4A^2\alpha_0^2 m$, the metric function has an inflexion point and the dynamic will be accelerated at late time if $3 < 4A^2\alpha_0^2 m$ or decelerated if $3 > 4A^2\alpha_0^2 m$.
– When $\alpha_0 < 0$, the metric function has a maximum. Its behaviour is decelerated if $n > (3,4A^2\alpha_0^2 m)$. If $n < (3,4A^2\alpha_0^2 m)$, the dynamic is accelerated at both late and early times. If the value of n is between $n = 3$ and $n = 4A^2\alpha_0^2 m$, the dynamic is decelerated at early time when $3 < 4A^2\alpha_0^2 m$ and becomes accelerated whereas when $3 > 4A^2\alpha_0^2 m$, it is first accelerated and then decelerated at late time.

In all the applications one can prove that the behaviours of a'' at early and late times are continuous. The sign of a'' does not change between the late and early times because $(\alpha_{1,2} - \alpha)'$ vanish for only one value of ϕ in the intervals in which the parameters of the three theories and the scalar field are allowed to vary. If it was not the case, the sign of this last expression would vanish for, at least, two values of the scalar field.

In the next subsection we will talk about the second derivative of the metric function in t time. For the sake of simplicity (the sign of the second derivative can change more than twice in t time) we will not study the behaviour of these theories in the t time (qualitatively, only the sign of the second derivative changes). Moreover, to do this we must carry out numerical computations as we will see, that seems diverge from our goal, i.e. make a general study of the dynamic whatever the coupling function.

4.7.3 Study of $\ddot{a}$.

Here, when $\ddot{a}$ and the first derivative are positives one can speak about inflation. We have:

$$\frac{\ddot{a}}{a} = \left[\frac{a''}{a} - \frac{a'^2}{a^2} - \frac{a'}{a}(\frac{b'}{b} + \frac{c'}{c})\right](abc)^{-2} \tag{4.34}$$

The relations (4.7) and (4.22) imply:

$$\frac{\ddot{a}}{a}(abc)^2\phi^2 = \frac{1}{2}\frac{\omega_\phi}{(3+2\omega)^2}\frac{\phi}{A^2} - \alpha(\beta_0 + \gamma_0) \tag{4.35}$$

This is an equation of first degree for α. Its solution is:

$$\alpha_3 = \frac{1}{2}\frac{\omega_\phi}{(3+2\omega)^2}\frac{\phi}{A^2}(\beta_0+\gamma_0)^{-1} \tag{4.36}$$

We use equation (4.23) to write:

$$\alpha - \alpha_3 = \alpha_0 - \frac{1}{2}\frac{1}{A\sqrt{3+2\omega}} - \frac{1}{2}\frac{\omega_\phi}{(3+2\omega)^2}\frac{\phi}{A^2}(\beta_0+\gamma_0)^{-1} \tag{4.37}$$

Then, one has to solve $\alpha - \alpha_3 = 0$ for ϕ so that we can determine the sign of this last expression for different intervals of the scalar field. This is not an easy task and to study the theories of the last subsection, we would need numerical computation. In a general manner, to simplify the resolution, one can notice that equation (4.37) is a third degree equation for $(3+2\omega)^{-1/2}$. Then, $\ddot{a}$ is positive when $\beta_0+\gamma_0 > 0$ (< 0) if $\alpha - \alpha_3 > 0$ (< 0) and negative when $\beta_0+\gamma_0 > 0$ (< 0) if $\alpha - \alpha_3 < 0$ (¿0). When a theory tends toward General Relativity, i.e. $\phi \to \phi_{RG}$, the dynamic of the metric function will be decelerated if:

$$\lim_{\phi\to\phi_{RG}} \omega_\phi < 2A^2\alpha_0(\beta_0+\gamma_0)(3+2\omega)^2\phi^{-1} \tag{4.38}$$

Under this condition one can not get inflation at late time. Note that (4.38) has the same form as (4.28) except the introduction of the constant $\beta_0+\gamma_0$. This comes from the fact that in the t time, all the metric functions appear in each field equations. If we use the three coupling functions of subsection 4.7.2 with equation (4.37), one obtain complex expressions which need numerical investigations to find their zeros.

Since the presence of matter tends to slow down the expansion, one can hypothesize that (4.38) could be a sufficient (but not necessary) condition so that model with matter has a decelerated behaviour in the same circumstances, that is at late time when the theory tends towards a relativistic behaviour.

4.8 Conclusions.

From the form of the coupling function $\omega(\phi)$, we can deduce the qualitative behaviour of the metric functions. It depends on the sign of $d\phi/d\tau$, $d\omega/d\phi$ and the integration constants α_0, β_0, γ_0. We have studied two things : sign of the first and second derivatives of the metric functions.

For the first derivative, the main difficulty is to find the zeros of ω_ϕ. When $\omega(\phi)$ is a monotonous function of the scalar field, we have eight basic possible behaviours ({1}, {2}, {3}, {4}, {1'}, {2'}, {3'}, {4'}) for a metric function because $d\phi/d\tau$, $d\omega/d\phi$ and the corresponding integration constants can be positive or negative (2*2*2=8). When $\omega(\phi)$ has one or several extrema, the behaviour of the metric function is a succession of behaviours of types {i} + {i'}, {i} and {i'} being the number of two of the eight basic behaviours, one with $\omega_\phi > 0$ and the other with $\omega_\phi < 0$. For the behaviours of type {1}, {1'}, {4} and {4'}, a complementary condition has to be fulfilled so that the metric function a (b, c) has an extremum : the value $2\alpha_0$ ($2\beta_0$, $2\gamma_0$) has to be in the interval in which $d\phi/d\tau$ varies otherwise the metric function is monotone. Or equivalently, a time independent formulation of this condition will be that the value of the scalar field corresponding to $3+2\omega = (2\alpha_0 A)^{-2}$ ($(2\beta_0 A)^{-2}$, $(2\gamma_0 A)^{-2}$) have to belong to the interval in which ϕ varies.

For the second derivative of the metric functions in the τ time, if the condition (4.25) is fulfilled, the dynamic of the metric functions is always accelerated. If it is not the case, we have to examine, for the metric function a for instance, the sign of $\alpha_1 - \alpha$ and $\alpha_2 - \alpha$. If these expressions have the same sign, the second derivative of a is positive otherwise it is negative.

In the t time, the dynamic is accelerated if (4.35) is positive and decelerated otherwise. If moreover, the first derivative is positive, we have inflation.

With this method we have been able to completely determine, whatever τ, the qualitative form of the metric functions for three different theories. Each of them can be compatible with the value of the PPN parameters at late time if we adjust their parameters. By using the results of subsection 4.7.3 concerning the sign of the second derivative in the cosmic time and numerical calculations, it is also possible to obtain the qualitative form of the metric functions in the t time.

Moreover, if with $\omega \to +\infty$ and $\omega_\phi\omega^{-3} \to 0$, we want the three metric functions to be increasing and decelerated at late time in the cosmic time, we deduce of the study that we must have: $(\alpha_0,\beta_0,\gamma_0) > 0$ and A and ω_ϕ must have the same sign, which is positive since $\omega \to +\infty$ and $\omega_\phi < 2A^2 \inf[\alpha_0(\beta_0+\gamma_0)$, $\beta_0(\alpha_0+\gamma_0)$, $\gamma_0(\alpha_0+\beta_0)](3+2\omega)^2\phi^{-1}$ when ϕ tends towards ϕ_{RG}, ϕ_{RG} being the smallest value of the scalar field. In these conditions the metric functions have a power law form.

In section 4.6, we have determined the conditions to have 1, 2 or 3 increasing metric functions; in fact, this is a graphic translating of some information contained in the constraint equation of the field equations.

We have studied the simplest anisotropic cosmological model but we hope to extend this method to more complicated ones such as Bianchi types II and V and in more complex situations, i.e. with cosmological constant or potential. The main advantage of such study is to reveal completely the dynamic of the metric functions whatever the form of the coupling function and not only for a particular one or for asymptotic behaviour.

Chapitre 5

Dynamique asymptotique du modèle de Bianchi de type I: formalisme Lagrangien 2(1 article)

Dans ce chapitre, nous étudierons une théorie tenseur-scalaire définie par le Lagrangien

$$L = G(\phi)^{-1}R - \frac{\omega(\phi)}{\phi}\phi_{,\mu}\phi^{,\mu}$$

Comme dans le précédent chapitre, notre but sera de déterminer les signes des dérivées premières et secondes des fonctions métriques par rapport à G et ω. C'est donc une généralisation des résultats du chapitre 4, G étant cette fois une fonction inconnue du champ scalaire. Bien entendu, une transformation de ce champ scalaire, $\psi = G(\phi)^{-1}$, ramène la théorie définie ci-dessus à la théorie du chapitre précédent. Cependant pour être appliquée, il faut pouvoir inverser G ce qui n'est pas toujours analytiquement faisable. Aussi, si l'on veut obtenir des résultats aussi généraux que ceux du chapitre 4, il ne faut pas qu'ils dépendent d'une hypothètique inversion de G et l'on doit considérer le Lagrangien ci-dessus.
Nous appliquerons nos résultats à deux théories respectivement liées à la théorie de Brans-Dicke et à la théorie des cordes sans son tenseur antisymétrique et définie par

$$G^{-1} = e^{-\phi}$$
$$\omega = \omega_0\phi e^{-\phi}$$

et

$$G^{-1} = e^{-\phi} + n$$
$$\omega = \omega_0\phi(e^{-\phi} + n)$$

Là encore, nous ne sommes pas parvenu à généraliser cette méthode aux autres modèles de Bianchi ni à considérer une théorie à champ scalaire massif, c'est-à-dire avec une troisième fonction indéterminée du champ scalaire qui jouerait le rôle de son potentiel. Cette méthode semble donc atteindre là ses limites.

Dynamical study of the hyperextended scalar-tensor theory in the empty Bianchi type I model.

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Abstract

The dynamics of the hyperextended scalar-tensor theory in the empty Bianchi type I model is investigated. We describe a method giving the sign of the first and second derivatives of the metric functions whatever the coupling function. Hence, we can predict if a theory gives birth to expanding, contracting, bouncing or inflationary cosmology. The dynamics of a string inspired theory without antisymetric field strength is analysed. Some exact solutions are found.

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5.1 Introduction

We study the dynamics of the metric functions for the hyperextended scalar-tensor theory in the empty Bianchi type I model.

The cosmological principle is based on the hypothesis of an isotropic and homogeneous Universe. However, at early times, Universe could have been anisotropic. We can quote several reasons in favour of this hypothesis [60]. Firstly, the isotropic hypothesis rests on observations such as the cosmological background. But it does not rule out the possibility of an anisotropic Universe for primordial time. Secondly, if the Universe is too isotropic and homogeneous, it is difficult to explain formation of structures, like galaxies: presence of anisotropies is necessary. Last, it could be easier to avoid singular Universe under these conditions.

Anisotropic Universes are described by the Bianchi models. Among these models, the only ones which isotropize and are in accordance with our present Universe at late time, are these of type I, V, VII_0, VII_h and IX. Current observations favour open and flat models and recent measurements seem to indicate that our present Universe undergoes inflation [9, 10]. Then it is a serious possibility that our Universe be spatially flat. It corresponds to the Bianchi type I model which will be the geometrical framework of this paper.

An important field of study in cosmology is the introduction of scalar fields in gravity theories. There are many reasons to justify their presence. Firstly, they are predicted by unified theories and could be the result of the compactification of extra dimensions appearing in theories like supersymetric, Kaluza-Klein or string theories. Secondly, they provide a way to get inflation [27], ending naturally without any fine-tuning. At last, the scalar-tensor theories can respect the solar system tests [61] as well as nucleosynthesis one but make very different predictions from General Relativity at early time. Among the scalar tensor theories, the most famous and simplest generalisation of General Relativity is the Brans-Dicke theory [7]. The coupling between the graviton and the dilaton, represented by the scalar field ϕ, is described by a coupling constant ω. If it is larger than 500, the theory respects the solar system tests. However, string theory in the low energy limit, which could describe the physics of the early Universe, is identical to Brans-Dicke theory with $\omega = -1$ after scalar field redefinition. Such contradiction between these two values of the coupling constant looks like the cosmological constant (Λ) problem: its observed value is about 120 orders smaller than what expected from a theoretical point of view. One way to solve this problem is to choose a variable cosmological constant. We can adopt the same solution concerning the coupling constant and consider a coupling function depending on the scalar field, $\omega(\phi)$. Such theories are called Generalised Scalar-Tensor theories (GST) and have been studied in the FLRW [52, 62, 63, 64] and anisotropic [29, 65] models in presence of matter.

In these theories ϕ^{-1} plays the role of a varying gravitational constant. However such a choice seems to be arbitrary. It is interesting to consider a function $G(\phi)^{-1}$ instead of ϕ in front of the scalar curvature term in the Lagrangian: this is the Hyperextended Scalar-Tensor theory (HST) [35, 66]. It can be rewritten as a GST [31, 67] by redefining a scalar field $\Phi = G(\phi)^{-1}$. Then we need to find the inverse function of $G(\phi)^{-1}$ which is not always analytically defined. This justifies the study of the HST.

Let's write few words about the relations between GST and HST and their relationship with General Relativity. The GST are agreed with the solar system tests if at late time $\omega \rightarrow \infty$ and $\omega_\phi \omega^{-3} \rightarrow 0$. For the HST, there is an additional unknown function of the scalar field, $G(\phi)^{-1}$. If we put $\Phi = G(\phi)^{-1}$, we obtain a GST with a coupling function written $\Omega(\phi)$. It can be expressed as a function of $\omega(\phi)$ and $G(\phi)^{-1}$: $\Omega(\Phi) = \omega(\phi)G(\phi)^{-1}(G_\phi^{-1})^{-2}\phi^{-1}$. Then, we deduce that the two conditions so that HST is agreed with solar system tests will be respectively: $\omega G^{-1}(G_\phi^{-1})^{-2}\phi^{-1} \rightarrow \infty$ and $(G_\phi^{-1})^3 G^2 \omega^{-2}\phi^2(\omega_\phi \omega^{-1} + G_\phi^{-1}G - \phi^{-1} - 2G_{\phi\phi}^{-1}G) \rightarrow 0$. If we choose $G(\phi)^{-1} = \phi$, we recover the usual conditions for GST. Lots of gravitation theories belong to HST class as dilaton gravity with $G^{-1} = 1/2e^{-\phi}$ and $\omega = -1/2\phi e^{-\phi}$, generally coupled scalar field with $G^{-1} = 1/2(\gamma - \xi\phi^2)$ and $\omega = 1/2\phi$, induced gravity with $G^{-1} = 1/2\epsilon\phi^2$ and $\omega = 1/2\phi$, etc [68]. It is difficult to choose physically interesting G^{-1} and ω. Different periods of the Universe could be approximated by different coupling functions. A way to select them is to impose that the theory be in accordance with the solar system tests at late time. We can also use dynamical criterions: the metric functions should be increasing at late time, eventually have a minimum so that they avoid the Big Bang, and have an inflationary period.

It is in view of determining such characteristics for the metric functions that we will examine the dynamics of the HST in the empty. A more realistic model will take into account matter fields. But then, most of time only asymptotic studies are workable for a given form of ω and G^{-1}. Generally it does not allow to detect the presence of several extrema, quasi-static phases for the dynamics or multiple inflationary phases. Our motivation is also to detect such physically important behaviours for any form of ω and G^{-1}, i.e. to study the dynamics of the metric functions whatever the value of the time and not only asymptotically. The price to pay for this full description of the dynamics is the absence of matter fields.

However, since their presence tends to oppose to expanding Universe, we hope that necessary and sufficient conditions we will establish to get expansion, inflation or quasi-static phases for instance in an empty model, will be either necessary or sufficient if matter fields are present. Hence, more complete studies of large classes of new theories specified by ω and G^{-1} with matter fields could be stimulated if they already have physically interesting dynamical characteristics in the empty. At the opposite, large classes of theories could be discriminated if their dynamical behaviours in the empty were in contradiction with current observations.

The paper is organised as follows: in section 5.2, we write the field equations in the empty Bianchi type I model and introduce new variables to transform them into a differential system of first order. We give the exact solution of the field equations. In section 5.3, we study the sign of the first derivatives of the metric functions and determine in which conditions they are increasing, decreasing or have extrema. In section 5.4, we study their second derivatives to predict the appearance of inflation or quasi-static phases. In these two last sections, we applied our results to a string inspired theory without H-field. We conclude in section 5.5 by showing the advantages of the method we present in this work to study any empty HST. We give the conditions on $G(\phi)^{-1}$ and $\omega(\phi)$ so that the Universe respects the solar system tests, be in expansion and accelerated at late time, and avoid the Big-Bang.

5.2 Field equations and exact solution

5.2.1 Field equations

We use the following line element:

$$ds^2 = -dt^2 + e^{2\alpha}(\omega^1)^2 + e^{2\beta}(\omega^2)^2 + e^{2\gamma}(\omega^3)^2 \tag{5.1}$$

where the ω^i:

$$\omega^1 = dx$$
$$\omega^2 = dy$$
$$\omega^3 = dz$$

are the 1-forms of the Bianchi type I model, t the proper time and e^α, e^β, e^γ the metric functions depending on t. The Lagrangian of the HST is written:

$$L = G(\phi)^{-1}R - \frac{\omega(\phi)}{\phi}\phi_{,\mu}\phi^{,\mu} \tag{5.2}$$

G and ω depend on the scalar field and specify the theory. Varying the action with respect to the space-time metric and scalar field, we obtain the field equations and the Klein-Gordon equation:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = G\left[\frac{\omega}{\phi}\phi_{,\mu}\phi_{,\nu} - \frac{\omega}{2\phi}\phi_{,\lambda}\phi^{\lambda}g_{\mu\nu} + (G^{-1})_{,\mu;\nu} - g_{\mu\nu}\Box(G^{-1})\right] \tag{5.3}$$

$$\dot{\phi}^2\left[-\frac{\omega_\phi}{\phi} + \frac{\omega}{\phi^2} - G(G^{-1})_\phi\frac{\omega}{\phi}\right] + \frac{2\omega}{\phi}\Box\phi + 3G(G^{-1})_\phi\Box(G^{-1}) = 0 \tag{5.4}$$

a dot meaning a derivative with respect to t time. Using the form (5.1) of the metric and τ time defined by $dt = e^{\alpha+\beta+\gamma}d\tau$, we get:

$$\alpha'' + \alpha' G(G^{-1})' + \tfrac{1}{2}G(G^{-1})'' = 0 \tag{5.5}$$

$$\beta'' + \beta' G(G^{-1})' + \tfrac{1}{2}G(G^{-1})'' = 0 \tag{5.6}$$

$$\gamma'' + \gamma' G(G^{-1})' + \tfrac{1}{2}G(G^{-1})'' = 0 \tag{5.7}$$

$$\alpha'\beta' + \alpha'\gamma' + \beta'\gamma' + G(G^{-1})'(\alpha' + \beta' + \gamma') - \omega\tfrac{G}{2}\tfrac{\phi'^2}{\phi} = 0 \tag{5.8}$$

$$\phi'^2\left[-\frac{\omega_\phi}{\phi} + \frac{\omega}{\phi^2} - G(G^{-1})_\phi\frac{\omega}{\phi}\right] - 2\omega\frac{\phi''}{\phi} - 3G(G^{-1})_\phi(G^{-1})'' = 0 \tag{5.9}$$

a prime meaning a derivative with respect to τ. The functions α, β and γ play equivalent roles in the field equations. So, in what follows, we will only consider the metric function e^α.

We are interested in the signs of first and second derivatives of the metric functions and not in their amplitudes. Since the product $e^{\alpha+\beta+\gamma}$ is positive, the signs of the first derivatives in the τ and t times will be the same whereas they will be different for second derivatives. Hence, to determine the sign of $(e^\alpha)^\cdot$, we will study this of α' in section 5.3. In section 5.4 we will determine the signs of the second derivatives by studying separately $(e^\alpha)^{\cdot\cdot}$ and $(e^\alpha)''$. This is justified by the fact that sometimes solutions are known in the τ time and not in the t time.

Now, we define new variables A, B, C and F in order to transform the second order field equations into a first order system:

$$\begin{aligned} A &= \alpha' G^{-1} \\ B &= \beta' G^{-1} \\ C &= \gamma' G^{-1} \\ F &= \tfrac{1}{2}(G^{-1})' \end{aligned} \tag{5.10}$$

Then, after integration, the spatial components of the field equations are written:

$$A + F = A_0 \tag{5.11}$$

$$B + F = B_0 \tag{5.12}$$

$$C + F = C_0 \tag{5.13}$$

A_0, B_0 and C_0 being integration constants. We also integrate the Klein-Gordon equation and get:

$$\frac{3}{4}(G^{-1})'^2 + \frac{1}{2}G^{-1}\omega\phi^{-1}\phi'^2 = -\Pi \tag{5.14}$$

Π being an integration constant. This last relation is written again:

$$\left[\frac{3}{4}(G^{-1})_\phi^2 + \frac{G^{-1}\omega}{2\phi}\right]\phi'^2 = -\Pi \tag{5.15}$$

From the constraint equation of the field equations we deduce the following relation between the integration constants:

$$A_0B_0 + A_0C_0 + B_0C_0 = -\Pi \tag{5.16}$$

The quantity between square brackets in the left hand-side of equation (5.15) is proportional and has the same sign as the energy density of the scalar field in the Einstein frame. For physical reasons, we will take a positive energy density, i.e.

$$\frac{3}{4}(G^{-1})_\phi^2 + \frac{G^{-1}\omega}{2\phi} > 0 \tag{5.17}$$

Hence, we deduce that $-\Pi > 0$. If we choose $G^{-1} = \phi$, we recover the usual relation for a positive energy density for GST, i.e. $3 + 2\omega > 0$. The sign of ϕ' is constant and depends on the sign of the square root of the energy density: if we take it positive (negative), the scalar field will be increasing (decreasing). Hence, the scalar field being a monotone function of time, it will be considered as a time variable.

From now, we will just consider the first spatial component of the field equations, i.e. equation (5.11) since we only need to study the dynamics of e^{α}. The set of values (A,F), solution of (5.11), can be graphically represented in the (A,F) plane by a straight line. During time evolution, the dynamics of the solution is described by the motion of a point of coordinate (A,F) on this set.

5.2.2 Exact solution

Using (5.11) and the first relation of (5.10), we deduce the exact solution for $\alpha(\tau)$:

$$\alpha - \alpha_0 = \int \frac{A_0}{G^{-1}} d\tau - \frac{1}{2} ln(G^{-1}) \tag{5.18}$$

α_0 being an integration constant. If we write $d\tau = \phi'^{-1} d\phi$ and express ϕ' using (5.15), we obtain $\alpha(\phi)$:

$$\alpha - \alpha_0 = \int \frac{A_0}{G^{-1}} \sqrt{-\frac{1}{\Pi}\left[\frac{3}{4}(G^{-1})_\phi^2 + \frac{G^{-1}\omega}{2\phi}\right]} d\phi - \frac{1}{2} ln(G^{-1}) \tag{5.19}$$

and analogous relations for β and γ with couples of constants (β_0,B_0) and (γ_0,C_0) respectively instead of (α_0,A_0).

There are two interesting asymptotical values for the couple (A,F). The first one is $(A,F) \rightarrow (0,A_0)$. It means that $G^{-1} \rightarrow 2(A_0\tau + A_1)$. Then, we deduce from (5.18) that the metric function tends toward a constant. Thus, the point $(0,A_0)$ stands for the static solution for e^{α}. The second one is $(A,F) \rightarrow (A_0,0)$. Then G^{-1} tends toward a constant. From (5.18) we get that $\alpha \rightarrow \alpha_1\tau + \alpha_2$, α_1 and α_2 being some constants. The function β and γ will behave in the same way in respectively $(B_0,0)$ and $(C_0,0)$. In the t time, this solution for the metric functions corresponds to power laws of t.

5.3 First derivatives of the metric functions

Using (5.19), we can write α' as a function of ϕ and then study its sign. However, even in the case of very simple functions G^{-1} and ω, the expression thus obtained is often difficult to analyse. The method we describe below allow to get in a simple manner the sign of the first derivative.

5.3.1 Sign of the first derivative

Now, we are explaining how to determine the sign of the first derivative of α for successive intervals of scalar field, considered as a time variable. For the clarity of the discussion, we will assume that ϕ is an increasing function of t or τ time, i.e. $\sqrt{-\Pi} > 0$. Moreover, we will need to evaluate $(G^{-1})'$ and $(G^{-1})''$ for some values of the scalar field. To this end, we express the derivatives of G^{-1} with respect to τ as functions of ϕ. Since $(G^{-1})' = G^{-1}_\phi \phi'$, we obtain:

$$(G^{-1})' = (G^{-1})_\phi \sqrt{\frac{-\Pi}{\frac{3}{4}(G^{-1}_\phi)^2 + \frac{G^{-1}\omega}{2\phi}}} \tag{5.20}$$

In the same way, we get:

$$(G^{-1})'' = -4\Pi \frac{2(G^{-1})_{\phi\phi}\omega G^{-1}\phi - (G^{-1})_\phi^2\omega\phi + (G^{-1})_\phi(\omega G^{-1} - G^{-1}\omega_\phi\phi)}{(2G^{-1}\omega + 3\phi(G^{-1})_\phi^2)^2} \tag{5.21}$$

To apply our method we need also to determine the following intervals:

1. The scalar field variation interval is defined by the condition (5.17): its energy density in the Einstein frame have to be positive. We write it as $[\phi_0,\phi_n]$.
2. We split it in several subintervals such as in each of them, G^{-1}, $(G^{-1})'$ and $(G^{-1})''$ have constant signs. We note these subintervals $[\phi_0,\phi_n] = [\phi_0,\phi_1] \cup ... [\phi_{l-1},\phi_l] \cup ... \cup [\phi_{n-1},\phi_n]$.

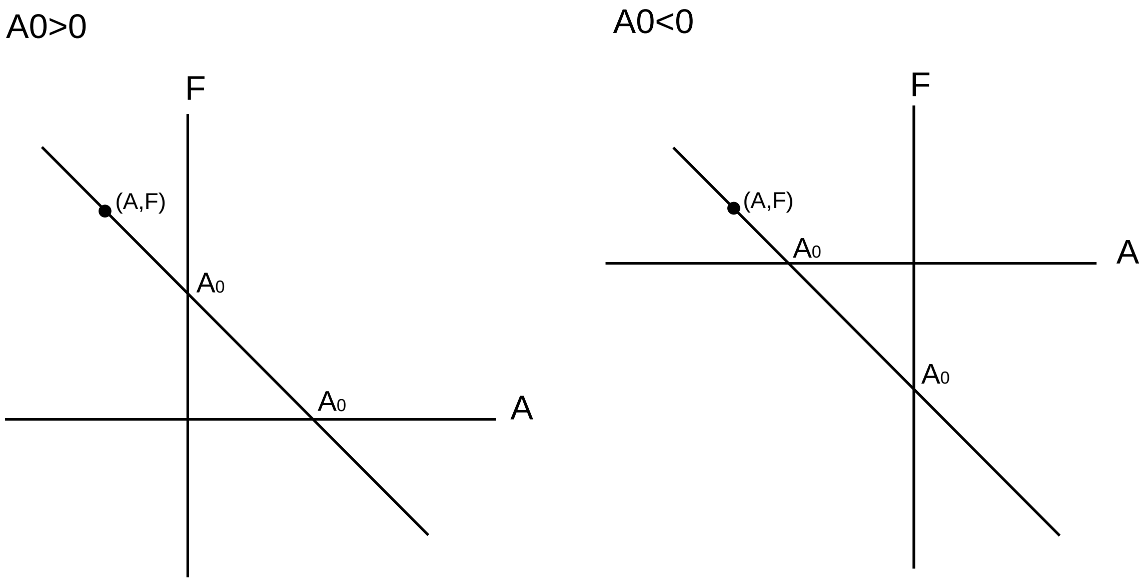

FIG. 5.1 – *The straight line describing the set of solutions (A,F) of the first spatial component of the field equations in the plane (A,F). To know the sign of the first derivative of the metric function e^{α}, we have to analysed the dynamics of a point (A,F) on this straight line. For that, we split the scalar field variation interval, considered as a time variable, in n sub-intervals such as the signs of G^{-1}, $(G^{-1})'$ and $(G^{-1})''$ be constant. Hence, on each of this interval $[\phi_{l-1},\phi_l]$, we know the sign of F given by $(G^{-1})'$ and in which direction the point (A,F) moves on the straight line depending on the fact that F increases or decreases, i.e. on the sign of $(G^{-1})''$. Then we have to check if F can take the value $2A_0$ when $\phi \in [\phi_{l-1},\phi_l]$. When it is true, it means that a metric function have an extremum for this range of the scalar field. Otherwise, it is monotone. Thus we deduce what is the sign of A on this scalar field interval and, as G^{-1} has also a constant sign, we get the sign of $\alpha' = AG$. Hence, on each interval $[\phi_{l-1},\phi_l]$, we obtain the sign of the first derivative of the metric function.*

Remember that they can be compared to time intervals since ϕ is an increasing function of time.
As a first step, we have to determine the direction of the motion of the point (A,F) on the straight line defined by equation (5.11) (see figure 5.1). Since $F = 1/2(G^{-1})'$, it means that on each interval $[\phi_{l-1},\phi_l]$ when $(G^{-1})'' > 0$, F increases and thus the point (A,F) moves from the right to the left. Otherwise, F decreases and the points moves from the left to the right.
As a second step, we determine the sign of A on each interval $[\phi_{l-1},\phi_l]$. Lets illustrate this point when $A_0 > 0$:

- If $(G^{-1})' < 0$, F is negative. We see on the straight line represented on figure 5.1 that then $A > 0$ whatever the sign of $(G^{-1})''$.
- If $(G^{-1})' > 0$ and $(G^{-1})'' > 0$, F is positive and increases on $[\phi_{l-1},\phi_l]$: $F \in \left[1/2(G^{-1})'(\phi_{l-1}),1/2(G^{-1})'(\phi_l)\right]$. Since the sign of A changes when $F = A_0$, we have to check if this value belongs or not to this last interval. We have three possibilities:
  - If $(G^{-1})'(\phi_l) < 2A_0$, it implies that $(G^{-1})'(\phi_{l-1}) < 2A_0$ and then $A > 0$.
  - If $(G^{-1})'(\phi_{l-1}) > 2A_0$, it implies that $(G^{-1})'(\phi_l) > 2A_0$ and then $A < 0$.
  - If $(G^{-1})'(\phi_{l-1}) < 2A_0$ and $(G^{-1})'(\phi_l) > 2A_0$, as F increases, first we have $A > 0$ and then $A < 0$.
- If $(G^{-1})' > 0$ and $(G^{-1})'' < 0$, F is positive and decreases. Then, for the same reasons as before, we have three possibilities:
  - If $(G^{-1})'(\phi_{l-1}) < 2A_0$, it implies that $(G^{-1})'(\phi_l) < 2A_0$ and then $A > 0$.
  - If $(G^{-1})'(\phi_l) > 2A_0$, it implies that $(G^{-1})'(\phi_{l-1}) > 2A_0$ and then $A < 0$.
  - If $(G^{-1})'(\phi_{l-1}) > 2A_0$ and $(G^{-1})'(\phi_l) < 2A_0$, as F decreases, first we have $A < 0$ and then $A > 0$.

Hence, this shows that the sign of A when the point (A,F) moves on the straight line of figure 5.1 representing the solution of the equation (5.11), is perfectly determined on each interval $[\phi_{l-1},\phi_l]$ by the sign of $(G^{-1})'$, $(G^{-1})''$ and the ordering of the quantities $(G^{-1})'(\phi_{l-1})$, $(G^{-1})'(\phi_l)$ and $2A_0$. Of course, the method is the same if the scalar field is decreasing or $A_0 < 0$.
As a third and last step, we determine the sign of α' on each intervals $[\phi_{l-1},\phi_l]$. Since the signs of A and G^{-1} are known on $[\phi_{l-1},\phi_l]$, we immediately deduce the sign of $\alpha' = AG$: If $G > 0$ ($G < 0$), when $A > 0$, the metric function is increasing (decreasing). Otherwise, it is decreasing (increasing).

Thus, on each interval $[\phi_{l-1},\phi_l]$ we are able to determine if the metric function is increasing, decreasing or have an extremum. The scalar field being a monotone function of time, we can describe for any time the evolution of the sign of the first derivative of e^α, i.e. its dynamics.

What happens when the theory respects the solar system tests? Our present Universe being in expansion, we write the conditions so that a metric function is increasing depending on G^{-1}, ω and their derivatives with respect to ϕ. Since $A = A_0 - F = \alpha' G^{-1}$, the metric function is increasing on an interval of scalar field if $(A_0 - 1/2(G^{-1})')G > 0$. Moreover, we know that $\Omega = \omega G^{-1}(G^{-1}_\phi)^{-2}\phi^{-1}$ have to diverge at late time so that the theory is compatible with the relativistic values of the PPN parameters. If we examine the relation (5.20), we deduce that this limit corresponds to $(G^{-1})' \rightarrow 0$. Hence, for a theory respecting the solar system tests at late time, the metric function α will be increasing if $A_0 G > 0$. Since gravitation is attractive and G can play the role of an effective gravitational constant, we have $G^{-1} > 0$ and thus $A_0 > 0$. Finally, as $(G^{-1})' = 2F \rightarrow 0$, we deduce also that all the metric functions tend toward power law in the t time as shown previously.

To summarise, for an expanding Universe, respecting the solar system tests at late time, all the metric functions tends toward power laws of the proper time and the initial conditions are such as $(A_0,B_0,C_0) > (0,0,0)$. Mathematically, it would be interesting to transform the system of equations (5.11-5.13) so that we use the dynamical system methods and analyse if power laws solutions for the metric functions correspond to future attractor. Such a study is beyond the scope of this paper and will be done in a next one.

In what follows, it seems to be physically reasonable to impose $G^{-1} > 0$. For the GST, it is equivalent to choose $\phi > 0$.

5.3.2 Applications

Brans-Dicke theory

We chose:

$$G^{-1} = e^{-\phi}$$
$$\omega = \omega_0 \phi e^{-\phi}$$

with $\omega_0 > -3/2$ so that the energy density of the scalar field is positive. This choice corresponds to the string theory without H-field and with the term ω_0 in front of $\phi_{,\mu}\phi^{,\mu}$ instead of 1. By putting $\Phi = e^{-\phi}$, we recover the Lagrangian of the Brans-Dicke theory with a coupling constant equal to $-\omega_0$. We calculate that:

$$(G^{-1})'' = 0$$
$$(G^{-1})' = -2\sqrt{-\Pi/(3+2\omega_0)}$$

The sign of α' is the same as A since $G^{-1} > 0$. F is a negative constant equal to $F_0 = -\sqrt{-\Pi/(3+2\omega_0)}$. If $F_0 > A_0$ then (A,F) is such as $A < 0$ and the metric function is decreasing. It is increasing otherwise (figure 5.2). We recover the usual dynamics of the Brans-Dicke theory for the Bianchi type I model.

String inspired theory

We modify the previous Lagrangian. In string theory, we can take into account the string loop effects by substituting the coupling function $e^{-\phi}$ by the series $B(\phi) = e^{-\phi} + a_0 + a_1 e^{\phi} + a_2 e^{2\phi} + ...$ In our application, we will limit the series to its two first terms [69, 70]. Note that no theory predicts the value of the a_i today. Hence, cosmological applications are susceptible to restrict the range of these parameters. Moreover, we consider again that ω_0 can be different from 1. This is justified by the fact that in our four dimensional Universe, it could exist moduli fields whose forms depend on the compactification scheme, producing $\omega_0 \neq 1$ [71]. Recent progress have been made on dual transformations concerning empty string theory (i.e. without any axion or moduli fields) with a constant ω_0 [72].
We examine the string inspired theory without H-field defined by:

$$G^{-1} = e^{-\phi} + n$$
$$\omega = \omega_0 \phi (e^{-\phi} + n) \tag{5.22}$$

Using (5.19), we have calculated the exact solution of the field equations (see 5.6). It is clearly easier to use the method described above than derive the sign of α' from this solution. As ϕ is increasing, the theory will

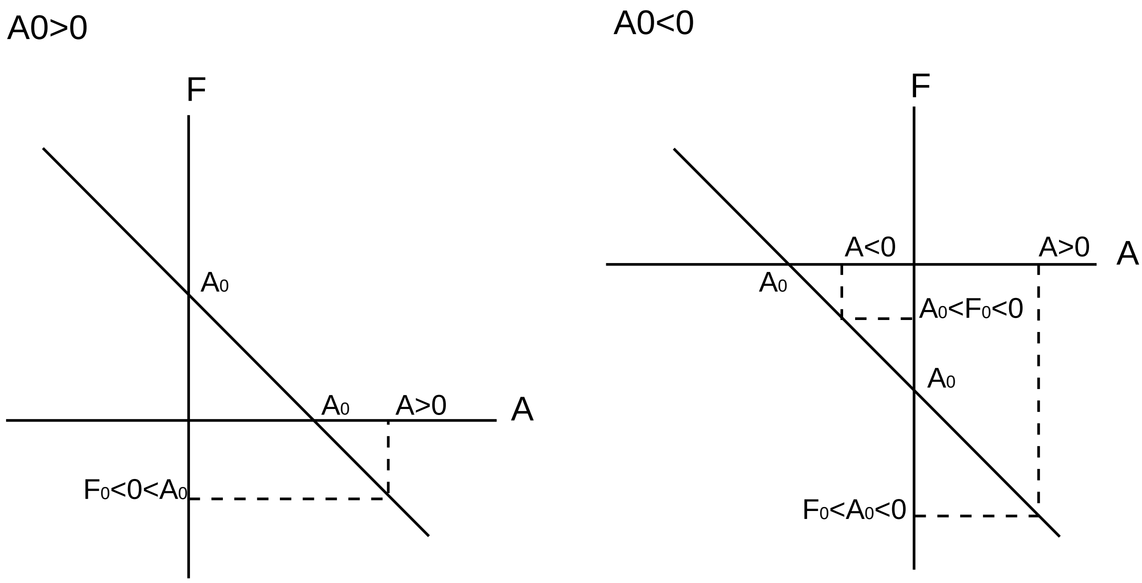

FIG. 5.2 – *The Brans-Dicke theory. We have sign of $(G^{-1},(G^{-1})',(G^{-1})'') = (+,-,0)$ for any value of ϕ. F is a negative constant equal to F_0. Whatever the sign of A_0, when $F < A_0$, $A > 0$. If $A_0 < 0$, when $F > A_0$, then $A < 0$. The sign of A is the same as the first derivative of e^α.*

TAB. 5.1 – *Scalar field variation intervals such as its energy density and G^{-1} be positives*

	$\omega_0 < -3/2$	$\omega_0 \in [-3/2,0]$	$\omega_0 > 0$
$n < 0$	$\phi \in [\phi_2, ln(-1/n)]$	$\phi \in]-\infty, ln(-1/n)]$	$\phi \in]-\infty, ln(-1/n)]$
$n > 0$	energy density < 0	$\phi \in]-\infty, \phi_1]$	$\phi \in]-\infty, +\infty[$

respect the solar system tests at late time for $\phi \to +\infty$. Then $G^{-1} \to n$ and n can be seen as the present value of the gravitational constant.

We search for the scalar field variation interval so that G^{-1} and its energy density is positive. This last quantity vanishes for $e^{\phi_{1,2}} = 1/n(-1 \pm \sqrt{-3/(2\omega_0)})$. After few algebra we obtain the table 5.1 giving all the possible scalar field variation intervals depending on n and ω_0.

We find that the sign of $(G^{-1})'$ is always negative and conclude that $F < 0$ whatever ϕ. The sign of $(G^{-1})''$ is the same as $n\omega_0$: it means that $2F = (G^{-1})'$ is a monotone function. Hence, the signs of $(G^{-1},(G^{-1})',(G^{-1})'')$ are these of $(+,-,n\omega_0)$: they are constant whatever ϕ and we have no need to split the scalar field variation interval in n sub-intervals.

This last result and the "step 2", show that we have to compare the values of $(G^{-1})'$ to the constant $2A_0$ when ϕ is equal to the boundaries of each of its variation interval so that we detect the presence or absence of extrema. We calculate that:
$(G^{-1})'(+\infty, -\infty, \phi_{1,2}, -ln(-n)) = (0, -\sqrt{-4\Pi(3+2\omega_0)^{-1}}, -\infty, -\sqrt{-4\Pi/3})$.

From these results, we use the method described in section 5.3 to get the dynamics of the metric function e^α:

- If $A_0 > 0$:
  - As $2F = (G^{-1})' < 0$, A is always positive. Since $G^{-1} > 0$, it follows that the metric function e^α is always increasing.
- If $A_0 < 0$:
  - If $\omega_0 < -3/2$ and $n < 0$, $2F = (G^{-1})'$ increases from $-\infty$ to $-\sqrt{-4\Pi/3}$. If this last value is smaller than $2A_0$, e^α is increasing. Otherwise it has a maximum. This case is shown on figure 5.3.
  - If $\omega_0 \in [-3/2,0]$ and $n > 0$, $(G^{-1})'$ decreases from $-\sqrt{-4\Pi/(3+2\omega_0)}$ to $-\infty$. If the first value is smaller than $2A_0$, e^α is increasing. Otherwise, it has a minimum.
  - If $\omega_0 \in [-3/2,0]$ and $n < 0$, $2F = (G^{-1})'$ increases from $-\sqrt{-4\Pi/(3+2\omega_0)}$ to $-\sqrt{-4\Pi/3}$. If the two values are smaller than $2A_0$, e^α is increasing. If both are larger than $2A_0$, e^α is decreasing. If $2A_0$ belongs to the interval defined by these values, e^α has a maximum.
  - If $\omega_0 > 0$ and $n > 0$, $2F = (G^{-1})'$ increases from $-\sqrt{-4\Pi/(3+2\omega_0)}$ to 0. If the first value is larger than $2A_0$, e^α is decreasing, otherwise a maximum exists. It is the only case for which

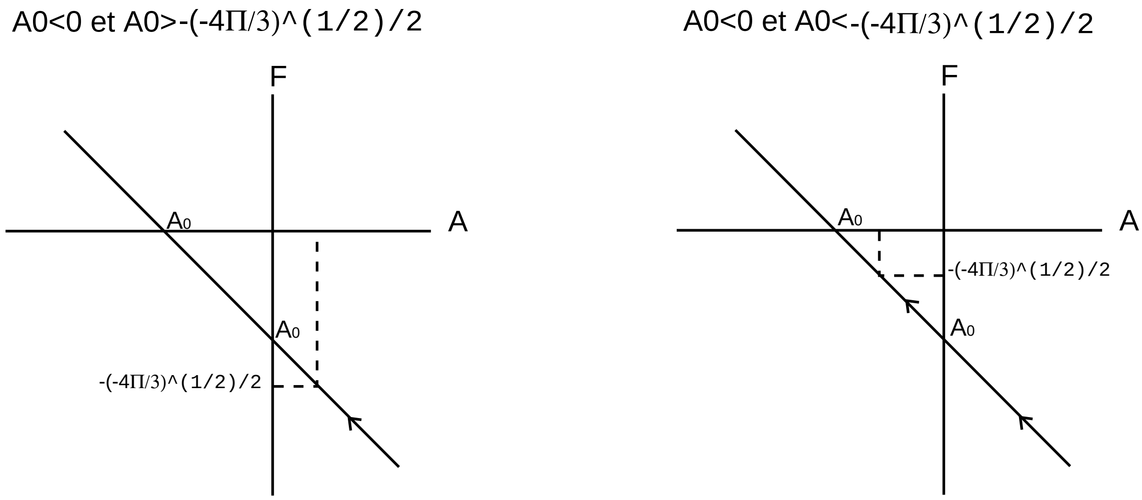

FIG. 5.3 – *The string inspired theory when $\omega < -3/2$, $n < 0$ and $A_0 < 0$. F increases (this is indicates by the direction of the arrows on each straight line of the figures) from $-\infty$ to $-\sqrt{-4\Pi/3}/2$ since $(G^{-1})'' > 0$. On the first figure, this last value is smaller than A_0. Then A is always positive and since $G^{-1} > 0$, α' is positive. The metric function e^α is increasing. On the second figure, $-\sqrt{-4\Pi/3}/2$ is larger than A_0. As long as $F < A_0$, $A > 0$ and then when $F > A_0$, $A < 0$. Since A and α' have the same sign we deduce that the metric function has a maximum.*

at late time, the metric function tends toward a power law type for t. Moreover, the theory is compatible with solar system tests since $\phi \to +\infty$. However, the dynamics at late time is not in accordance with the observations. On this simple example, we see that conditions for the respect of the solar system tests are not sufficient to ensure a realistic dynamics of the Universe for our present time.

– If $\omega_0 > 0$ and $n < 0$, $2F = (G^{-1})'$ decreases from $-\sqrt{-4\Pi/(3+2\omega_0)}$ to $-\sqrt{-4\Pi/3}$. If these two values are smaller than $2A_0$, e^α is increasing. If they are larger than $2A_0$, e^α is decreasing. If $2A_0$ belongs to the interval defined by these values, it has a minimum.

Hence the method described in the previous section allows to know all the conditions for which the metric functions are decreasing, increasing or "bouncing". It would have been more difficult to get the same results from the exact solution $\alpha(\phi)$.

5.4 Sign of the second derivative

In this section, we study the sign of the second derivatives of e^α in τ and t times. This is justified by the fact that sometimes solutions are known in one time but not in the other. In what follows, we assume that we know the scalar field variation interval.

5.4.1 Sign of the second derivative in the τ time

The sign of the second derivative of the metric function in the τ time is the same as:

$$G^{-2}(e^\alpha)'' = G^{-2}e^\alpha(\alpha'' + \alpha'^2) \tag{5.23}$$

The spatial component (5.5) of the field equations provides:

$$G^{-2}\alpha'' = -A(G^{-1})' - 1/2G^{-1}(G^{-1})'' \tag{5.24}$$

From the equation (5.11) we get:

$$G^{-2}\alpha'^2 = A^2 = (A_0 - 1/2(G^{-1})')^2 \tag{5.25}$$

Then, from the two last equations we deduce that the sign of $(e^\alpha)''$ is the same as:

$$G^{-2}(\alpha'' + \alpha'^2) = \frac{3}{4}(G^{-1})'^2 - 2A_0(G^{-1})' - \frac{1}{2}G^{-1}(G^{-1})'' + A_0^2 \tag{5.26}$$

It is a second-degree equation for $(G^{-1})'$. With the help of the relations (5.20-5.21), we can express its coefficients as a function of the scalar field. Its determinant is equal to:

$$\begin{aligned}\Delta &= A_0^2 + 6\Pi G^{-1}[-G^{-1}\omega(G^{-1})_\phi + \phi\omega(G^{-1})_\phi^2 + \phi G^{-1}(G^{-1})_\phi\omega_\phi - \\ &\quad 2\phi G^{-1}\omega(G^{-1})_{\phi\phi}]/(2G^{-1}\omega + 3\phi(G^{-1})_\phi^2)^2 \end{aligned} \tag{5.27}$$

and its roots are:

$$(G^{-1})'_{root\frac{1}{2}} = \frac{4A_0 \pm \sqrt{\Delta}}{3} \tag{5.28}$$

We deduce that:

- If $\Delta < 0$, the second-degree equation is negative and then $(e^\alpha)'' > 0$.
- If $\Delta > 0$, $(G^{-1})' - (G^{-1})'_{root1}$ and $(G^{-1})' - (G^{-1})'_{root2}$ have different signs, $(e^\alpha)'' < 0$.
- If $\Delta > 0$, $(G^{-1})' - (G^{-1})'_{root1}$ and $(G^{-1})' - (G^{-1})'_{root2}$ have the same signs, $(e^\alpha)'' > 0$.

All these inequalities can be expressed as some functions of ϕ. From them, it is possible to derive the scalar field intervals so that $(e^\alpha)''$ is positive or negative. Since we can determine the scalar field intervals for which the sign of $(e^\alpha)'$ is constant, it is possible to describe completely the dynamical evolution of the metric function $\alpha(\tau)$.

5.4.2 Sign of the second derivative in the t time

The sign of the second derivative in the t time is the same as:

$$\frac{d^2e^\alpha}{dt^2} = [(e^\alpha)'' - (e^\alpha)'(\alpha' + \beta' + \gamma')]\, e^{-2(\alpha+\beta+\gamma)} \tag{5.29}$$

Using (5.23) to express $G^{-2}(e^\alpha)''$ and the relations (5.20-5.21), we get the expression giving the sign of $(e^\alpha)^{\cdot\cdot}$:

$$\begin{aligned} G^{-2}\frac{d^2e^\alpha}{dt^2}e^{\alpha+2(\beta+\gamma)} &= (Bo+Co)\left[-Ao + \frac{(G^{-1})_\phi}{2}\sqrt{-\frac{\Pi}{\frac{G^{-1}\omega}{2\phi} + \frac{3(G^{-1})_\phi^2}{4}}}\right] \\ &\quad -2\Pi G^{-1}[-G^{-1}\omega(G^{-1})_\phi + \phi\omega(G^{-1})_\phi^2 + \phi G^{-1}(G^{-1})_\phi\omega_\phi \\ &\quad -2\phi G^{-1}\omega(G^{-1})_{\phi\phi}]/(2G^{-1}\omega + 3\phi(G^{-1})_\phi^2)^2 \end{aligned} \tag{5.30}$$

Since it can be written as a function of the scalar field, it is possible to deduce the scalar field intervals so that $(e^\alpha)^{\cdot\cdot}$ is positive or negative. By Comparing them with these for which the sign of the first derivative is constant, we will get the qualitative dynamical behaviour of $\alpha(t)$. When the sign of the second derivative of the metric function with respect to t is positive on a scalar field interval, the dynamics is accelerated. If, at the same time [1], the metric function is increasing, we are in the presence of inflation. Lets note that it happens naturally without any potential. Such phenomenon has been studied in the GST and received the name of kinetic inflation [27]. Inflation in the HST has been studied in [51]. When the right hand side of the equation (5.30) vanishes, the metric function e^α have a point of inflection. Physically, it means that we could be in presence of a quasi-static phase for the dynamics of the Universe, at least in the direction associated with the metric function e^α.

If we assume that the theory is in agreement with solar system tests at late time, then we know that $\Omega \to \infty$ and $(G^{-1})' \to 0$. We introduce this limit in the expression (5.30) and obtain the condition to have an inflationary behaviour for the metric function at late time: $2\Pi G^{-1}[-G^{-1}\omega(G^{-1})_\phi + \phi\omega(G^{-1})_\phi^2 + \phi G^{-1}(G^{-1})_\phi\omega_\phi - 2\phi G^{-1}\omega(G^{-1})_{\phi\phi}]/(2G^{-1}\omega + 3\phi(G^{-1})_\phi^2)^2 < -A_0(B_0 + C_0)$. If our present Universe undergoes inflation (observations of higher redshift objects seem to be necessary to confirm this phenomenon [73]), this last inequality could play the same role as the two conditions necessary so that a GST respects the solar system test (i.e. $\omega > 500$ and $\omega_\phi\omega^{-3} \to 0$) and thus, help to select physical interesting HST.

1. Here, we consider ϕ as the time variable.

5.4.3 Application in the t time: the string inspired theory

For this theory, (5.30) takes the form:

$$4\Pi n\omega_0 e^{\phi}(1+ne^{\phi})^2(2\omega_0 n^2 e^{2\phi}+4\omega_0 ne^{\phi}+3+2\omega_0)^{-2}+ (B_0+C_0)\left[-A_0-\sqrt{-2\Pi(2\omega_0 n^2 e^{2\phi}+4\omega_0 ne^{\phi}+3+2\omega_0)^{-1}}\right] \quad (5.31)$$

We look for the asymptotic sign of this equation for the different cases described in table 5.1 and depending on n and ω_0. From this table, we deduce:

- If $\omega_0 < -3/2$ and $n < 0$, e^{α} is decelerated at early time. At late time the second derivative has the sign of $-(B_0+C_0)(A_0+\sqrt{-\Pi/3})$.
- If $\omega_0 > -3/2$ and $n < 0$, the second derivative at early and late times has respectively the sign of $-(B_0+C_0)(A_0+\sqrt{-\Pi/(2\omega_0+3)})$ and $-(B_0+C_0)(A_0+\sqrt{-\Pi/3})$.
- If $\omega_0 \in [-3/2{,}0]$ and $n > 0$, the second derivative at early time has the sign of $-(B_0+C_0)(A_0+\sqrt{-\Pi/(2\omega_0+3)})$. The dynamics of the metric function is accelerated at late time. Since we have shown that it is always increasing for these values of ω_0 and n, we have inflation.
- If $\omega_0 > 0$ and $n > 0$, the second derivative at early time has the sign of $-(B_0+C_0)(A_0+\sqrt{-\Pi/(2\omega_0+3)})$ whereas at late time it has this of $-A_0(B_0+C_0)$.

Lets note that if $A_0 > 0$, e^{α} is always an increasing function and any accelerated behaviour will correspond to inflation. Moreover, (5.31) can be seen as a polynome for e^{ϕ}. We do not make its complete study since this section is just an application but it seems to be clear that it should have more than one zero. Hence, the theory should have several phases of inflation. Mathematically, an asymptotical study could not have detected such behaviour. This is one of the advantage of the method presented in this paper.

5.5 Conclusion

We have studied the dynamical behaviour of the metric functions for the HST in the empty Bianchi type I model for any form of G^{-1} and ω. Such dynamical study has always been done for the GST with matter field in FLRW models [52, 62, 63, 64] and Bianchi models [29, 65]. However, most of time it concerns asymptotic behaviours. Here, we have made the choice to consider a simpler theory, i.e. without matter field, but to study its dynamics for any time and not only asymptotically. Mathematically, we get a more accurate description of the dynamics than with asymptotical methods. The spliting of the scalar field variation interval in several ones allow to get all the extrema of the metric functions as well as their types. The calculation of the zeros of equation (5.30) enable to get the intervals of ϕ, considering as a time variable, in which a metric function is accelerated, decelerated as well as its inflexion points. Comparing these two types of scalar field intervals, we are able to describe completely the dynamical behaviour of the metric functions. Thus physically, it is possible to predict if a theory, defined by $\omega(\phi)$ and $G(\phi)$, will give birth to an Universe with several bounces. Such a scenario could be one of the keys to homogenise the Universe in the manner of a Mixmaster model. We can also predict if there will have several periods of inflation. It is also an interesting behaviour since some problems need inflation to be solved (age problem, isotropisation) whereas for others, it is prefered that the Universe be decelerated (formation structures). Last, we can detect quasi-static phases which are likely to favour the appearance of some structures we observe in the Universes and to solve the age problem. We think that the detection of such characteristics in an empty model may stimulate and justify more complex researches when matter fields are present. Asymptotical studies are generally not able to detect such behaviours.

We have applied this method to a string inspired theory. Since we have determined the exact solution of the field equations as function of ϕ (cf 5.19), it is easy to calculate the exact solution of this theory (5.6). Clearly, it seems to be difficult to study the dynamical behaviour of the metric function from the expression thus obtained. We have shown that the late time behaviour of this theory is not compatible with our observed Universe. However it could be an interesting model for early time behaviour. The metric functions are monotone or have one and only one extremum. If all the metric functions have a minimum, the Big-Bang can be avoided. For that, it is necessary that $\omega_0 \in [-3/2{,}0]$ and $n > 0$ or $\omega_0 > 0$ and $n < 0$. We have also shown that several periods of inflation are possible.

Although GST is often claimed to be equivalent to HST it is only true if the inverse function of G^{-1} can be analytically determined [51]. Hence, HST is a more general class of scalar tensor theories than GST which is obtained for $G^{-1} = \phi$. In this case, it is well known that the theory will respect the solar system tests if $\omega \to \infty$ and $\omega_\phi \omega^{-3} \to 0$. For any form of G, these conditions become $\omega G^{-1}(G^{-1}_\phi)^{-2}\phi^{-1} \to \infty$

and $(G_\phi^{-1})^3G^2\omega^{-2}\phi^2(\omega_\phi\omega^{-1}+G_\phi^{-1}G-\phi^{-1}-2G_{\phi\phi}^{-1}G)\rightarrow 0$. For this limit, we have shown that the metric functions tend toward a power law type in the t time and G^{-1} toward a constant which then may correspond to the present value of the gravitational constant. Mathematically, it would be interesting to study the fields equation (5.11-5.13), which are first order equations, in the light of the dynamical system methods [25] so that we learn if this behaviour could be a late time attractor. Physically, the fact that G tends toward a constant is associated with a power law types for the metric functions is in good agreement with what should be the dynamical behaviour of our present Universe and thus confirmed the viability of scalar tensor theories.

We conclude by giving the conditions on G and ω so that the Universe at late time, respects the solar system tests, be accelerated and bouncing. When the theory respects the solar system tests, we have $(G^{-1})'\rightarrow 0$. Then, the Universe is in expansion and accelerated if A_0G is positive, $2\Pi G^{-1}[-G^{-1}\omega(G^{-1})_\phi+\phi\omega(G^{-1})_\phi^2+\phi G^{-1}(G^{-1})_\phi\omega_\phi-2\phi G^{-1}\omega(G^{-1})_{\phi\phi}]/(2G^{-1}\omega+3\phi(G^{-1})_\phi^2)^2<-A_0(B_0+C_0)$ and other similar conditions obtained by circular permutations on A_0, B_0 and C_0. Since G could play the role of an effective gravitational constant, it means that today $G>0$ and thus $(A_0,B_0,C_0)>0$. This restricts the range of the initial conditions. If moreover we want that all the metric functions have a minimum so that the Big-Bang is avoided and if we assume that G was positive (negative) at early time, we need to choose $G(\phi)^{-1}$ and $\omega(\phi)$ so that $(G^{-1})''$ is negative (positive). The conditions concerning the respect of the solar system tests and these described in this paragraph and concerning the dynamics of the metric functions put strong constraints on the form of $G(\phi)^{-1}$ and $\omega(\phi)$. For the latter, to our knowledge, we have not seen equivalent ones in the literature.

5.6 Appendix: Exact solution of the string inspired theory

From (5.19), we get $\alpha(\phi)$:

Solution with $\omega_0>0$:

$$\alpha-\alpha_0= \tag{5.32}$$

$-ln(\sqrt{e^{-\phi+n}})+Ao\{\sqrt{2}\sqrt{\omega_0}\phi+\sqrt{3}ln(e^{-\phi}+n)-\sqrt{3+2\omega_0}ln\{-\Pi(3+2\omega_0+2e^\phi n\omega_0+e^\phi\sqrt{3+2\omega_0}[(3+2\omega_0+4e^\phi n\omega_0+2e^{2\phi}n^2\omega_0)/(e^{2\phi}-\Pi)]^{1/2}\sqrt{-\Pi})/e^\phi\}-\sqrt{3}ln\{-3-\Pi e^{-\phi}+\sqrt{3}[(3+2\omega_0+4e^\phi n\omega_0+2e^{2\phi}n^2\omega_0)/(e^{2\phi}-\Pi)]^{1/2}(-\Pi)^{3/2}\}+\sqrt{2}\sqrt{\omega_0}ln\{-\omega_0\Pi e^{-\phi}+-n\omega_0\Pi+-\Pi\sqrt{\omega_0/2}[(3+2\omega_0+4e^\phi n\omega_0+2e^{2\phi}n^2\omega_0)/(e^{2\phi}-\Pi^{3/2})]^{1/2}\}\}/(2\sqrt{-\Pi})$

with $\tau-\tau_0=\sqrt{-\Pi}(6\phi+2\sqrt{6}\sqrt{\omega_0}arctan\{(3+2\omega_0+2e^\phi n\omega_0)/(\sqrt{6}e^\phi n\sqrt{\omega_0})\}+3ln\{(3+2\omega_0+4e^\phi n\omega_0+2e^{2\phi}n^2\omega_0)e^{-2\phi}\})/(3n^2\omega_0)$.

Solution with $\omega_0<0$:

$$\alpha-\alpha_0= \tag{5.33}$$

$-ln(\sqrt{e^{-\phi}+n})+Ao\{\sqrt{-\omega_0}arctan\{(\sqrt{2}e^\phi(1+e^\phi n)\sqrt{-\omega_0}[(3-2-\omega_0-4e^\phi n-\omega_0-2e^{2\phi}n^2-\omega_0)/(-\Pi e^{2\phi})]^{1/2}\sqrt{-\Pi})/(-3+2-\omega_0+4e^\phi n-\omega_0+2e^{2\phi}n^2-\omega_0)\}/(\sqrt{-2\Pi})+\sqrt{3}ln(e^{-\phi}+n)/(2\sqrt{-\Pi})+((-3+2-\omega_0)ln\{-\Pi(3-2-\omega_0-2e^\phi n-\omega_0+e^\phi\sqrt{3-2-\omega_0}[(3-2-\omega_0-4e^\phi n-\omega_0-2e^{2\phi}n^2-\omega_0)/(-\Pi e^{2\phi})]^{1/2}$
$\sqrt{-\Pi})e^{-\phi}\})/(2[-\Pi(3-2-\omega_0)]^{1/2})-\sqrt{3}ln\{-3-\Pi e^{-\phi}+\sqrt{3}[(3-2-\omega_0-4e^\phi n-\omega_0-2e^{2\phi}n^2-\omega_0)/(-\Pi e^{2\phi})]^{1/2}-\Pi^{3/2}\}/(2\sqrt{-\Pi})\}$.

with $\tau-\tau_0=\sqrt{-\Pi}(-6\phi+2\sqrt{6}\sqrt{-\omega_0}arctanh\{(-3-2\omega_0-2e^\phi n\omega_0)/(\sqrt{6}e^\phi n\sqrt{-\omega_0})\}-3ln\{(-3-2\omega_0-4e^\phi n\omega_0-2e^{2\phi}n^2\omega_0)e^{-2\phi}\})/(-3n^2\omega_0)$

Chapitre 6

Dynamique asymptotique du modèle de Bianchi de type I: formalisme Hamiltonien(1 article)

Ce chapitre présente notre première utilisation du formalisme Hamiltonien ADM. On souhaite déterminer sous quelles conditions un champ scalaire conduit un modèle de Bianchi de type I à s'isotropiser asymptotiquement tout en étant en expansion et avec un potentiel positif, dans les référentiels de Brans-Dicke et d'Einstein. On analyse alors les deux cas particuliers pour lesquels les fonctions métriques tendent vers des lois en puissance ou en exponentielle du temps propre dans le référentiel d'Einstein.
L'avantage du formalisme Hamiltonien sur le formalisme Lagrangien est que le système d'équations obtenu est du premier ordre et est donc plus facile à analyser. L'inconvénient, c'est que les résultats sont évidement exprimés en fonction des variables Hamiltoniennes dont l'interprétation physique n'est pas toujours aussi commode que celles du formalisme Lagrangien. Le formalisme Hamiltonien sera l'un des ingrédients principaux que nous utiliserons dans la partie IV de cette thèse lorsque nous analyserons le processus d'isotropisation des modèles cosmologiques de Bianchi.

Hamiltonian study of the Generalized scalar-tensor theory with potential in a Bianchi type I model

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Abstract

We study the generalized scalar tensor theory with a potential in the Bianchi type I model by using the ADM formalism. We examine the conditions for the Universe to be in expansion, isotropic and with a positive potential at late time in the Brans-Dicke and Einstein frames. In particular, we analyse the two important cases where metric functions tend, in an asymptotic way, toward power or exponential laws in the Einstein frame.

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6.1 Introduction

We study the Generalized Scalar Tensor theory with a potential depending on a scalar field in the Bianchi type I model. This theory has the same form as Brans-Dicke theory but with a coupling function depending on a scalar field. Its dynamical behaviour with matter, but without a potential, in the homogeneous Bianchi type models has been studied by Wands and Mimoso [29] and in the FLRW models by Barrow and Parson [52].

The potential can be considered as an effective cosmological constant. Such a constant can rule out the Universe age problem [74]. The cosmological constant is a source of negative pressure able to accelerate the expansion of Universe and hence to give birth to inflation. Then, the Universe would seem younger than it is. Moreover, new observations [9][10] would show that the Universe is undergoing inflation and this the presence of a positive cosmological constant although this accelerated behaviour remains to be confirmed. However the present value on this constant is in contradiction with the value predicted by particle physics for the early Universe. This is the reason why a model with a varying effective cosmological constant is so interesting. One recalls that the empty generalized scalar tensor theory can naturally generate inflation without a potential: this is what is usually called kinetic inflation [28][27].

The aim of this work is to analyse under which conditions the Universe can isotropize and be in expansion with a positive potential at late time in the Einstein and Brans-Dicke frames. Once they are derived, we look for additional conditions such that the metric functions tend asymptotically toward exponential or power laws of the proper time in the Einstein frame. We discuss whether such theories can respect the solar system tests. When no matter field is present, this means that the coupling function ω becomes infinite or at least greater than 500 and $\omega_\phi \omega^{-3}$ tends to vanish, where ω_ϕ is the derivative of ω with respect to the scalar field. No such conditions are known in a theory with a potential but if we add one and consider it as an effective cosmological constant, the observations show that it should be rather small at late time. So it seems reasonable to assume that these three conditions, $\omega \to \infty$, $\omega_\phi \omega^{-3} \to 0$ and a small potential at late time, are necessary but not sufficient for the solar system tests to be respected in a generalized scalar tensor theory with a potential. Note that even if a generalized scalar tensor theory tends toward a relativistic behaviour, it does not mean that its solutions, in these conditions, will tend toward relativistic one as shown in [75].

To obtain these results, we will employ a Hamiltonian formalism and more precisely the ADM formalism. It is often used in quantum cosmology, to find the wave-function of the Universe but less so to study classical problems such as the search for exact solutions or dynamics of the classical field equations [76]. Usually Lagrangian methods are preferred.

This paper is organised as follows: in section 6.2, we establish the field equation of the ADM formalism in the Einstein frame. In section 6.3 we analyse the dynamics of the theory in this frame and when it isotropizes. In section 6.4, we examine which conditions have to be respected by the Hamiltonian and the scalar field so that the Universe can isotropize and be in expansion at late times in the Brans-Dicke frame with a positive potential. In section 6.5, we discuss the best conditions in each frame so that the Universe can be isotropic, expanding, and with positive potential at late times and say a few words about the production of exact solutions. Using these elements, we look for the conditions such that the metric functions of the generalized scalar tensor theory tend toward exponential or power law solutions in the Einstein frame.

6.2 Field equations

In the Einstein frame, the metric can be written as:

$$ds^2 = -(\bar{N}^2 - \bar{N}_i\bar{N}^i)d\bar{\Omega}^2 + 2\bar{N}_i d\bar{\Omega}\omega^i + R_0^2 e^{-2\bar{\Omega}} e^{2\beta_{ij}}\omega^i\omega^j \tag{6.1}$$

the ω^i being the 1-forms of the Bianchi type I model. The barred quantities are those of the Einstein frame. $\bar{N}$ and $\bar{N}_i$ are respectively the lapse and shift functions. The relation between the metric functions of the Einstein and Brans-Dicke frames is:

$$g_{ij} = \bar{g}_{ij}\phi^{-1} \tag{6.2}$$

With $(i,j) = 0,1,2,3$. Hence, in the Brans-Dicke frame, a potential U of the Einstein frame can be written as:

$$U_{BD} = U\phi^2 \tag{6.3}$$

The Lagrangian of the generalized scalar tensor theory with a potential is given by:

$$S = (16\pi)^{-1}\int \left[\bar{R} - (3/2 + \omega(\phi))\phi^{,\mu}\phi_{,\mu}\phi^2 - U(\phi)\right]\sqrt{-\bar{g}}d^4x \tag{6.4}$$

where ϕ is a positive scalar field, $\omega(\phi)$ is the coupling function, and $U(\phi)$ is the potential. As the Universe is homogeneous, the scalar field depends on time variable only. We use the method employed in [77][78] to find the ADM Hamiltonian. The ADM form of the action is written as:

$$S = (16\pi)^{-1}\int(\pi^{ij}\frac{\partial\bar{g}_{ij}}{\partial\bar{t}} + \pi^\phi\frac{\partial\phi}{\partial\bar{t}} - \bar{N}C^0 - \bar{N}_iC^i)d^4x \tag{6.5}$$

the π^{ij} and π^ϕ are, respectively, the conjugate momentum of the metric functions $\bar{g}_{ij}$ and the scalar field, $\bar{N}$ and $\bar{N}_i$ play the role of Lagrange multipliers. The quantities C^0 and C^i are, respectively, the super-Hamiltonian and the supermomentum defined by:

$$C^0 = -\sqrt{^{(3)}\bar{g}}^{(3)}\bar{R} - \frac{1}{\sqrt{^{(3)}\bar{g}}}(\frac{1}{2}(\pi_k^k)^2 - \pi^{ij}\pi_{ij}) + \frac{1}{\sqrt{^{(3)}\bar{g}}}\frac{\pi_\phi^2\phi^2}{6+4\omega} + \sqrt{^{(3)}\bar{g}}U(\phi) \tag{6.6}$$

$$C^i = \pi^{ij}_{|j} \tag{6.7}$$

the "$^{(3)}$" hold for the quantities calculated on the 3-space and the "$|$" for the covariant derivative in the 3-space. By varying the action with respect to $\bar{N}$ and $\bar{N}^i$ we find the two constraints $C^0 = 0$ and $C^i = 0$. Then, by using them and the form of the metric functions, $\bar{g}_{ij} = R_0^2 e^{-2\bar{\Omega}} e^{2\beta_{ij}}$ with $(i,j) = 1,2,3$, and after taking the surface integral $\int\omega^1\wedge\omega^2\wedge\omega^3$ equal to $(4\pi)^2$ [1], the action (6.5) becomes:

$$S = 2\pi\int \pi_k^i d\beta_{ik} - \pi_k^k d\bar{\Omega} + 1/2\pi_\phi d\phi \tag{6.8}$$

The final form of the action is obtained by defining the traceless diagonal matrix β_{ij} and p_{ij} by following the procedure introducing by Misner [79]. We define:

$$p_k^i = 2\pi\pi_k^i - \frac{2}{3}\pi\delta_k^i\pi_l^l \tag{6.9}$$

and parameterise:

$$6p_{ij} = diag(p_+ + \sqrt{3}p_-, p_+ - \sqrt{3}p_-, -2p_+) \tag{6.10}$$

$$\beta_{ij} = diag(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+) \tag{6.11}$$

Moreover, on the hypersurface of constant time, $\pi^{ij}_{|j} = 0$ for the Bianchi type I and IX without rotation. Using the expression (6.9)-(6.11), the action (6.8) can be written as:

$$S = \int p_+d\beta_+ + p_-d\beta_- + p_\phi d\phi - Hd\Omega \tag{6.12}$$

1. This value is valuable for Bianchi type I and IX models.

with $p_\phi = \pi\pi_\phi$ and $H = 2\pi\pi^k_k$. We can obtain the expression of the quantity H, that is π^k_k from the constraint $C^0 = 0$. Then, we find for H:

$$H^2 = p_+^2 + p_-^2 + 12\frac{p_\phi^2\phi^2}{3+2\omega} + 36\pi^2 R_0^4 e^{-4\bar\Omega}(V-1) + 24\pi^2 R_0^6 e^{-6\bar\Omega}U \tag{6.13}$$

The potential $V(\beta_+,\beta_-)$ depends on the Bianchi model. For the Bianchi type I model, $V = 1$. Finally the field equations for the generalized scalar tensor theory are Hamilton's equations for the Hamiltonian H:

$$H^2 = p_+^2 + p_-^2 + 12\frac{p_\phi^2\phi^2}{3+2\omega} + 24\pi^2 R_0^6 e^{-6\bar\Omega}U \tag{6.14}$$

$$\dot\beta_\pm = \frac{\partial H}{\partial p_\pm} = \frac{p_\pm}{H} \tag{6.15}$$

$$\dot\phi = \frac{\partial H}{\partial p_\phi} = \frac{12\phi^2 p_\phi}{(3+2\omega)H} \tag{6.16}$$

$$\dot p_\pm = -\frac{\partial H}{\partial \beta_\pm} = 0 \tag{6.17}$$

$$\dot p_\phi = -\frac{\partial H}{\partial \phi} = -12\frac{\phi p_\phi^2}{(3+2\omega)H} + 12\frac{\omega_\phi \phi^2 p_\phi^2}{(3+2\omega)^2 H} - 12\pi^2 R_0^6 \frac{e^{-6\bar\Omega}U_\phi}{H} \tag{6.18}$$

$$\dot H = \frac{dH}{d\bar\Omega} = \frac{\partial H}{\partial\bar\Omega} = -72\pi^2 R_0^6 \frac{e^{-6\bar\Omega}U}{H} \tag{6.19}$$

where a dot denotes a derivative with respect to $\bar\Omega$. Moreover we will choose $\bar N^i = 0$ and we express $\bar N$ by writing that $\partial\sqrt{\bar g}/\partial\bar\Omega = -1/2\pi^k_k \bar N$ (see [78], p1830 for a detailed calculus). Hence, we find:

$$\bar N = \frac{12\pi R_0^3 e^{-3\bar\Omega}}{H} \tag{6.20}$$

Equation (6.17) shows that the conjugate momemta $p_\pm$ are constants. Then, from the equations (6.15), we deduce that $\beta_+ - (p_+ p_-^{-1})\beta_-$ is constant and the Universe point moves on a straight line in the (β_+,β_-) plane. Since we have $d\bar t = -\bar N d\bar\Omega$ 2, equation (6.20) shows that when the Hamiltonian has a constant sign, $\bar\Omega$ is a monotonous function of $\bar t$, decreasing if $H > 0$ and increasing otherwise.

6.3 Dynamical study of the metric functions in the proper time of the Einstein frame

In this section we analyse the dynamics of the metric functions in the Einstein frame. They can be written:

$$\bar g_{ij} = R_0^2 e^{-2\bar\Omega + 2\beta_{ij}} \tag{6.21}$$

With $(i,j) = 1,2,3$. Using $d\bar t = -\bar N d\bar\Omega$, we obtain:

$$\frac{d\bar{g_{ij}}}{d\bar t} = 2R_0^2\left(\frac{d\beta_{ij}}{d\bar t} - \frac{d\bar\Omega}{d\bar t}\right)e^{-2\bar\Omega+2\beta_{ij}} = 2R_0^2 e^{-2\bar\Omega+2\beta_{ij}}\frac{H - p_{ij}}{H\bar N} \tag{6.22}$$

the product $H\bar N$ being positive. We are interested in the sign of the quantity (6.22) which depends on the sign of $H - p_{ij}$. For sake of simplicity, we will consider a potential with a constant sign. We will see later how to extend our results to the case where the sign of the potential varies. Then, the equation (6.19) shows that the sign of $H\dot H$ is constant and so for H and $\dot H$. H is a monotonic function of time and p_{ij} is a constant, which means that equation (6.22) can only have one zero. So if there is an extremum for the metric function when the potential is of constant sign, it is unique. $\bar\Omega$ is also a monotonic function of $\bar t$.

If H_{ini} and H_{fin} are the two values of the Hamiltonian at the extremities of the $\bar t$ proper time interval, H will evolve monotonically from H_{ini} to H_{fin}. The first derivative (6.22) of the metric function in the Einstein frame will vanish if the three conditions C_1, C_2 and C_3 are true:

- C_1: H and p_{ij} have the same sign
- C_2 and C_3: p_{ij} belongs to the interval defined by H_{ini} and H_{fin}

2. We choose $d\bar t = -\bar N d\bar\Omega$ as in [77] but $d\bar t = \bar N d\bar\Omega$ is also a valid choice and would not change our results in t or $\bar t$ times.

Hence, we have to consider the following four cases for which we give the variation of the metric function depending on the $\bar{t}$ time:

Case 1a: $U < 0$, $\dot{H}$ and $H > 0$
We recall we have $d\bar{t} = -Nd\bar{\Omega}$. Hence taking into account (6.20), when $\bar{\Omega}$ increases, $\bar{t}$ decreases. The Hamiltonian is a decreasing function of $\bar{t}$. The three conditions C_i become:

- C_1: $p_{ij} > 0$
- C_2: $H_{fin} - p_{ij}$¡0
- C_3: $H_{ini} - p_{ij}$¿0

Whatever case we consider, if C_2 or C_3 are false, respectively C_3 or C_2 are true. In addition, in the present case, if C_3 is false, C_1 is true.

If the three conditions are true, the metric function has a maximum in the proper time of the Einstein frame since the Hamiltonian will be equal to p_{ij} for a value of $\bar{\Omega}$. If C_1 is wrong, it is increasing since then $H - p_{ij}$ has always the sign of H.

If C_2 is wrong, the Hamiltonian is always larger than p_{ij}, and the metric function is again increasing for the $\bar{t}$ time.

If C_3 is wrong, C_1 is true, and the metric function decreases since the Hamiltonian is always smaller than p_{ij}.

The same reasoning will hold for the other cases.

Case 1b: $U < 0$, $\dot{H}$ and $H < 0$
When $\bar{\Omega}$ increases, $\bar{t}$ is increasing. The Hamiltonian is a negative and decreasing functions of these times coordinates. When C_2 is wrong, C_1 is true. The three conditions can be written as:

- C_1: $p_{ij} < 0$
- C_2: $H_{fin} - p_{ij}$¡0
- C_3: $H_{ini} - p_{ij}$¿0

If they are all true, the metric function has a maximum.
If C_1 or C_3 is false, it is decreasing.
If C_1 is true, C_2 is false and it is increasing.

Case 2a: $U > 0$, $\dot{H} < 0$ and H >0
When $\bar{\Omega}$ increases, $\bar{t}$ decreases. The Hamiltonian is a positive and decreasing function of $\bar{\Omega}$ and then an increasing function of $\bar{t}$. When C_2 is wrong, C_1 is true. The three conditions can be written as:

- C_1: $p_{ij} > 0$
- C_2: $H_{fin} - p_{ij}$¿0
- C_3: $H_{ini} - p_{ij}$¡0

If they are all true, the metric function has a minimum.
If C_1 or C_3 is false, it is increasing.
If C_2 is false, C_1 is true, and the metric function is decreasing.

Case 2b: $U > 0$, $\dot{H} > 0$ and H <0
When $\bar{\Omega}$ increases, $\bar{t}$ increases. The Hamiltonian is a negative and increasing function of the two time coordinates. When C_3 is false, C_1 is true. We obtain for the three conditions:

- C_1: $p_{ij} < 0$
- C_2: $H_{fin} - p_{ij}$¿0
- C_3: $H_{ini} - p_{ij}$¡0

If the three conditions are true, the metric function has a minimum.
If C_1 or C_2 is false, it is decreasing.
If C_1 is true, C_3 is false, the metric function is increasing.

All these results are summarised in table 6.1.

Before analysing this table, lets note that we will use the expression ”Big-Bang singularity” to denote the fact that the three metric functions decrease toward zero. In addition the expression ”pancake singularity” or ”cigar singularity” apply, respectively, to the cases where one or two metric functions decrease toward zero.

	$H, \dot{H} > 0, U < 0$	$H, \dot{H}, U < 0$	$H, U > 0, \dot{H} < 0$	$\dot{H}, U > 0, H < 0$
C_1, C_2, C_3: true	Maximum	Maximum	Minimum	Minimum
C_1: false	Increasing	Decreasing	Increasing	Decreasing
C_2: false	Increasing			Decreasing
C_3: false		Decreasing	Increasing	
C_2: false, C_1: true		Increasing	Decreasing	
C_3: false, C_1: true	Decreasing			Increasing

TAB. 6.1 – *Dynamical behaviour of a metric function in the proper time of the Einstein frame depending on the signs of the potential, the Hamiltonian and its initial and final values.*

From the table 1, we obtain the following results in the Einstein frame. We deduce that a metric function could have a maximum (minimum) only in the presence of a negative (positive) potential. Moreover, all the conjugate momentum p_{ij} can not have the same sign and then the condition C_1 can not be true for all the metric functions. We deduce that, when the Hamiltonian is positive, the three metric functions can be increasing together at late times, but not decreasing. All types of singularity, Big-Bang type, pancake type or cigar type are possible at early time. When the Hamiltonian is negative, the three metric functions can be decreasing together at late time but not increasing. The singularity if it exists will only be of pancake or cigar type at early time. We have already written that as long as the potential has a constant sign, the metric function can have one and only one extremum. This is also the case when we consider flat or open FLRW models with trace-free matter, ϕ finite and $\omega_\phi > 0$ as shown in [52]. In this paper it is also proved that flat FLRW models can only contain a single minimum whereas here, a single maximum is also allowed for negative potential.

Lastly, it is easy to calculate that $d\beta_\pm/d\bar{t} \propto e^{3\bar{\Omega}}$. This means that the Universe will isotropize, that is $\bar{g}_{ij}/(d\bar{g}_{ij}/dt)$ tends toward the same function whatever i and j, only when $\bar{\Omega} \to -\infty$. This value will correspond to late (early) times for $\bar{t}$ if the Hamiltonian is positive (negative).

When the sign of the potential varies, the table is always true but H_{ini} and H_{fin} define the different intervals of values of the Hamiltonian for which the sign of the potential is constant. Hence, if asymptotically the sign of the potential is constant, one can always use the previous results.

6.4 Necessary and sufficient conditions to obtain an isotropic Universe in expansion at late time in the Brans-Dicke frame.

In this section we look for isotropisation and expansion of the metric functions at late time in the Brans-Dicke frame. Let t_0 be the maximum value (finite or not) of the t-time coordinate, that is the value of t at late time. We suppose that the physical conditions in t_0 are the same as those of today. Hence the scalar field will be such that $\omega > 500$, $\omega_\phi \omega^{-3} \to 0$ and $U \to 0$ or very small. These conditions have been assumed to be necessary so that the relativistic values of the PPN parameters are respected. In a Universe without any matter field [56][57] they can be written as:

$$\beta = 1 + O(\omega_\phi \omega^{-3}) \tag{6.23}$$

$$\gamma = 1 - (\omega + 2)^{-1} \tag{6.24}$$

Since in the generalized scalar tensor theory the inverse of the scalar field can be considered like the gravitational coupling function G, we assume that it tends toward a positive constant at late times. This is justified by measurements of the quantity $\dot{G}G^{-1}$. For a review of these experiments see [52]. Hence, a relativistic limit shall be asymptotically recovered.

A necessary and sufficient condition such that the Universe is isotropic at late time, that is $g_{ij}/(dg_{ij}/dt)$ tends toward the same function whatever i and j, will be:

$$d\beta_\pm/dt \propto e^{3\bar{\Omega}}\sqrt{\phi} \to 0 \tag{6.25}$$

that is $\beta_\pm$ tend toward a constant. Then, the three metric functions are proportional to the function $e^{-2\bar{\Omega}}$ in the Einstein frame or $e^{-2\bar{\Omega}}\phi^{-1}$ in the Brans-Dicke frame. If we want that the Universe be in expansion at late times, this last function have to be increasing in the Brans-Dicke frame when $t \to t_0$. We write the derivative of this function with respect to $\bar{\Omega}$:

$$(e^{-2\bar{\Omega}}\phi^{-1})^{\cdot} = -e^{-2\bar{\Omega}}\phi^{-1}(\frac{\dot{\phi}}{\phi} + 2) \tag{6.26}$$

There are two ways so that it can be increasing in the t-time. Firstly, we suppose that t_0 coincides with an infinite value of $\bar{\Omega}$.

If $\dot{\phi}\phi^{-1} > -2$ when $t \to t_0$, $e^{-2\bar{\Omega}}\phi^{-1}$ is a decreasing function of $\bar{\Omega}$ and it will be an increasing function of t if the Hamiltonian is positive. This means that $t \to t_0$ coincides with $\bar{\Omega} \to -\infty$. We note that for a decreasing scalar field on the t time there are no additional conditions coming from the fact that $\dot{\phi}\phi^{-1} > -2$ whereas an increasing one have to respect $\dot{\phi}\phi^{-1} \in [-2{,}0]$ when $t \to t_0$ [3]. Hence, in a Universe undergoing expansion at late times in the Brans-Dicke frame, increasing scalar field $\phi(t)$ implies fine-tuning.

As the scalar field tends toward a constant and $\bar{\Omega} \to -\infty$, equation (6.25) shows that the Universe isotropizes in a natural way, that is without any other condition.

If now $\dot{\phi}\phi^{-1} < -2$ when $t \to t_0$, $e^{-2\bar{\Omega}}\phi^{-1}$ is an increasing function of $\bar{\Omega}$ and it would be an increasing function of t if the Hamiltonian were negative. This means that $t \to t_0$ coincides with $\bar{\Omega} \to +\infty$. The scalar field is always a decreasing function of t.

The relation (6.25) shows that the Universe will isotropize at t_0 if then $\phi < e^{-6\bar{\Omega}}$.

Secondly, if we consider that the t_0 time coincides with a finite value of $\bar{\Omega}$, we can write the same conditions so that the function $e^{-2\bar{\Omega}}\phi^{-1}$ is an increasing function of t at late times depending on the sign of $\dot{\phi}\phi^{-1} + 2$, but to obtain an isotropic Universe the scalar field has to vanish in t_0 since from (6.25) we see that $d\beta_\pm/dt$ is now proportional to $\phi^{1/2}$.

Another fact to take into account to obtain a realistic Universe at late t time is the recently observed accelerated dynamics of the Universe which implies a positive cosmological constant. So that the potential, in Einstein or Brans-Dicke frames, is positive at late time, we deduce from (6.19) that when the Hamiltonian is positive (negative), it is a decreasing (increasing) function of $\bar{\Omega}$ and hence an increasing function of t.

Finally we summarise these results in table 2.

From the above, we deduce the following results in the Brans-Dicke frame. When $\bar{\Omega}$ diverges at late t-time, the Universe of the Bianchi type I model, in the generalized scalar tensor theory and in the Brans-Dicke frame, with a positive potential will isotropize and be in expansion if $\dot{\phi}\phi^{-1} > -2$ and the Hamiltonian is a positive and increasing function of the t-time. If $\dot{\phi}\phi^{-1} < -2$, the Hamiltonian have to be a negative and increasing function of the t-time and the scalar field has to be less than $e^{-6\bar{\Omega}}$. If $\bar{\Omega}$ tends toward a constant at late t-time, we need $\dot{\phi}\phi^{-1} > -2$ ($\dot{\phi}\phi^{-1} < -2$), a positive and increasing (negative and increasing) Hamiltonian in the t-time and a vanishing scalar field. Let us note, that a Universe able to isotropize at both late and early times can exist.

Remark: All the results of table 2 are expressed in the $\bar{\Omega}$-time except the sign of H and dH/dt. By defining the 3-Volume V in the Brans-Dicke time by $V = e^{-3\bar{\Omega}}\phi^{-3/2} = det\sqrt{^{(3)}g}\phi^{-3/2}$ one can also write the condition on the sign of $\dot{\phi}\phi^{-1} + 2$ with physical quantities of the Brans-Dicke time. By writing that $\dot{\phi} = \frac{d\phi}{dt}\frac{dt}{d\Omega}$, this expression becomes:

$$\frac{\dot{\phi}}{\phi} + 2 = \frac{d\phi}{dt}\phi^{-1}\left[ln(V^{-1/3}\phi^{1/2})\right]^{-1} \tag{6.27}$$

6.5 Discussions

Numerous works have been devoted to the problem of the physical frame between the Brans-Dicke or Einstein frame [80, 81, 82]. In the Brans-Dicke frame, the scalar field is related to the gravitational coupling function and is non-minimally coupled to the gravitational field. In the Einstein frame, the scalar field is associated with the rest mass of the particles and is minimally coupled to the gravitational field. Lets examine the optimal conditions in each frame to obtain asymptotically an isotropic expanding Universe with a positive potential.

In the Einstein frame, we need a positive Hamiltonian so that the three metric functions are increasing at late times. The potential will be positive if H is a decreasing function of $\bar{\Omega}$ and then an increasing one of $\bar{t}$. In these conditions, no more than two metric functions can have one and only one minimum. All types of singularity are possible at early time. Since the Universe isotropizes if $\bar{\Omega} \to -\infty$ and as $H > 0$, it will arise at late times.

3. We would have the inverse situation if we had considered a negative scalar field.

In the Brans-Dicke frame, an expanding and isotropic Universe with a positive potential at late times can be realized in four different ways described in table 2. However, only one of them does not need the scalar field to vanish asymptotically. It is such that the Hamiltonian has the same features as in the Einstein frame with $\bar{\Omega} \to -\infty$ and $\dot{\phi}/\phi > -2$. It is important to avoid the scalar field vanishing because it would mean that the gravitational constant is asymptotically infinite. However its current observed value seems to be small and constant.
Hence, this work does not allow us to argue in favour of one of the frame since the Hamiltonian and time $\bar{\Omega}$ have the same features in both frames, corresponding to the dynamics and properties of the Universe we want to obtain at late time, that is isotropy, expansion and positive potential. This is not a surprise because, when we have studied the late time behaviour of the metric functions in the Brans-Dicke frame, we have assumed that asymptotically the scalar field tended toward a constant. Thus, at late time, the two frames become similar.

Before carrying on with this discussion, it is useful to know how to find exact solutions from the system of equations (6.14)-(6.20) by making use of our previous results. In the generalized scalar tensor theory with a potential, two functions can be chosen arbitrarily to completely define the theory. The method we will use is the following:
We choose $U(\bar{\Omega})$ or $H(\bar{\Omega})$ and we determine respectively the Hamiltonian or the potential with (6.19) and then the functions $\beta_\pm$ with (6.15). Then, we choose $\omega(\bar{\Omega})$ or $\phi(\bar{\Omega})$ and with (6.14), we obtain respectively the scalar field or the coupling function. With the help of (6.20) and (6.2), we find $\bar{\Omega}(\bar{t})$ and $\bar{t}(t)$ and then the expressions of each quantity in the proper time of each frame. Since we have determined what are the characteristics of H, ϕ and $\bar{\Omega}$ to obtain physically interesting late time behaviour, it is easy to obtain as many exact solutions as we want with isotropic expanding behaviour and positive potential.
Other methods such as dynamical ones could be used to study the equations (6.14)-(6.20) since they constitute a system of first order differential equations. However our goal is to find conditions to obtain an asymptotically isotropic expanding Universe with a positive potential and here such a method is not necessary. Application of dynamical methods to the system of equations (6.14)-(6.20) will be the subject of future works. Some more powerful methods to derive exact solutions from Hamiltonian formalism have been developed in [76]. They rely on symmetries such as Killing tensor symmetries. However, it is difficult to predict the late time behaviour of the solutions thus obtained and, if they are very efficient when a perfect fluid is present, it is different if we consider any potential. The method explained above has the advantage of predicting the late time behaviour of the solution once the two unknown functions fixed thanks to the results of the previous sections. We will use it to examine two important asymptotical behaviours in the Einstein frame for the metric functions: exponential and power-law behaviours. We have chosen to study them in the Einstein frame rather than in the Brans-Dicke frame since we will be able to compare our results with those obtained in General relativity with a scalar field.

Firstly, we examine the exponential behaviour for the metric functions. Then, we shall try to recover the "No Hair Theorem" for the Bianchi type I model so that we test our results. Wald [49] has shown that, in the case of General Relativity with a scalar field and a cosmological constant, all the Bianchi models (except contracting Bianchi type IX) initially in expansion approach the isotropic De Sitter solution. If we consider the Generalized scalar tensor theory in the Einstein frame, we obtain a positive cosmological constant by choosing $H = \Lambda e^{-3\bar{\Omega}}$ with $\Lambda > 0$. Then, the metric functions are:

$$\bar{g}_{ij} = e^{\Lambda(6\pi R_0^3)^{-1}(\bar{t}-\bar{t}_0)+2p_{ij}(3\Lambda)^{-1}e^{-\Lambda(4\pi R_0^3)^{-1}(\bar{t}-\bar{t}_0)}+2\beta_{ij0}} \tag{6.28}$$

β_{ij0}, $\bar{t}_0$ and p_{ij} being some constants. $\bar{\Omega}$ varies from $+\infty$ to $-\infty$ and $\bar{t}$ respectively from $-\infty$ to $+\infty$. At late times, whatever the coupling function such that $\phi(\bar{\Omega})$ is defined for $\bar{\Omega} \to -\infty$, the Universe will isotropize and approach a De Sitter model in accordance with Wald. The properties of the Hamiltonian and the time $\bar{\Omega}$ correspond to those we have defined for this type of behaviour in section 6.3. At early time, when $\bar{\Omega} \to +\infty$, the $\beta_\pm$ functions dominate the dynamical behaviour, and the singularity will be of cigar or pancake type. If we choose $\Lambda < 0$ the behaviour of the late and early times are inverted.
Now, we make the opposite reasoning. We suppose that at late time, the 3-volume has an exponential behaviour. We want to know whether the Universe will isotropize and whether the potential and the coupling constant respect the solar system tests at late time. As we know the form of the 3-volume asymptotically, we can determine that of the Hamiltonian $H(\bar{\Omega})$ from $d\bar{t} = -\bar{N}d\bar{\Omega}$. With this expression, we can check that for a general asymptotical form $1/f(\bar{t})$ of the 3-volume, H^{-1} will be equal asymptotically to $(-12\pi R_0^3)^{-1}(\left[A(\bar{\Omega})G(\bar{\Omega})\right]^{\cdot} + \dot{B}(\bar{\Omega}))$, where A and B are any function such that $A \to 1$, $B \to 0$ when $\bar{\Omega} \to -\infty$, $G = F(f^{-1}(e^{3\bar{\Omega}}))$ and $F = \int f(\bar{t})d\bar{t}$. Here, $1/f(\bar{t}) = e^{3\bar{t}}$ and $G = -1/3e^{3\bar{\Omega}}$. We will

choose a class of Hamiltonian functions such that H and $\dot{H}$ do not oscillate at late times that is the first and second derivatives of A and B vanish asymptotically (however our results will be the same for types of functions such $\cos(\bar{\Omega}^{-1})$ and $\sin(\bar{\Omega}^{-1})$ corresponding respectively to A and B with damped oscillations and which have the same asymptotic characteristics described above). Hence, the Hamiltonian tends toward $e^{-3\bar{\Omega}}$. This form excludes any oscillating potential at late time. From (6.19), we deduce that $U \rightarrow C^2$, where C is a constant. It follows from the results of section 6.3, that if the scalar field is also defined in $\bar{\Omega} \rightarrow -\infty$, the Universe isotropize at late time, corresponding to this last value of $\bar{\Omega}$. Then it tends toward a De Sitter behaviour and the potential toward a positive constant. This generalizes the result of Wald for Bianchi type I model to any potential that is asymptotically constant and does not oscillate. Using (6.14) and (6.16), we show that asymptotically, $3+2\omega \propto \phi^2\dot{\phi}^2$ and $\omega_\phi\omega^{-3} \propto \dot{\phi}^4\phi^{-5} - \ddot{\phi}\dot{\phi}^2\phi^{-4}$. If ϕ tends toward a non-vanishing constant, then the coupling function and $\omega_\phi\omega^{-3}$ respectively diverge and vanishes asymptotically. This limit for the scalar field is the most interesting one since it is proportional to the inverse of the gravitational function. This leads to the fact that any ω satisfies the solar system tests for such a limit reached in $\bar{\Omega} \rightarrow -\infty$. It will also be the case for the potential if the constant C^2 is sufficiently small. Other limits for ϕ could be envisaged and not be in contradiction with the previous quoted tests or isotropisation in Einstein frame. Above all $\phi \rightarrow \infty$ which leads to an asymptotically vanishing gravitational constant. We will present some examples in the next paragraph.

Another interesting behaviour for the 3-volume is a power law one since more often we search for theories which tend toward General relativity at late times and since this last one, in isotropic and flat cases, has most of time power law solutions. Let us have a look at what happens when the 3-volume of the Universe tends toward a power law form of $\bar{t}$ at late time, that is $e^{-3\bar{\Omega}} \propto \bar{t}^{3m}$. We proceed in the same way as previously. We deduce that asymptotically the Hamiltonian tends toward $e^{l\bar{\Omega}}$ with $l = m^{-1} - 3$. To obtain a positive potential we shall have $l < 0$, that is $m \notin [0,1/3]$. Then, if the scalar field is defined in $\bar{\Omega} \rightarrow -\infty$, the Universe isotropizes and the metric functions tend toward a power law $\bar{t}^{2m}$. The Universe is in expansion if $m > 0$ that is $l > -3$ and undergoes inflation if $m > 1$, that is $l \in [-3,-2]$. In what it follows, we assume that l belongs to $[-3,0]$. From (6.19) we deduce that the potential is proportional to $e^{(2l+6)\bar{\Omega}}$. At late times, it vanishes in agreement with solar system tests, whatever l. Concerning the coupling function ω, we have the same limits as above as long as $l < 0$ and we can write the same things. However, here we shall also use the fact that $\dot{\phi} = \dot{U}U_\phi^{-1}$ and $\ddot{\phi} = \ddot{U}U_\phi^{-1} - \dot{U}^2U_{\phi\phi}U_\phi^{-3}$. However, at late time $\dot{U} = (2l+6)U$ and $\ddot{U} = (2l+6)^2U$. Thus, asymptotically $3+2\omega \propto \phi^2U_\phi^2U^{-2}$ and $\omega_\phi\omega^{-3} \propto U^3\left[UU_\phi + \phi(UU_{\phi\phi} - U_\phi^2)\right]U_\phi^{-5}\phi^{-5}$. So, for any given form of $U(\phi)$, we can determine whether the solar system tests will be recovered as the Universe isotropize in $\bar{\Omega} \rightarrow -\infty$. As an application, we examine two typical forms for the potential: $U = e^{k\phi}$ and $U = \phi^k$. For the first form, the scalar field shall tend toward $(2l+6)k^{-1}\bar{\Omega}$ and for the second one toward $e^{(2l+6)k^{-1}\bar{\Omega}}$. Both limits are defined for $\bar{\Omega} \rightarrow -\infty$. Thus, the forms we choose for the potential are compatible with a scalar field defined in $\bar{\Omega} \rightarrow -\infty$. Firstly, we examine $U = e^{k\phi}$ with $k > 0$. Such potentials are well motivated, especially from string theory. They are also used to generate scaling solutions for which the energy density of the scalar field mimics the equation of state of a barotropic fluid [83] although they are not necessary well adapted [84] to this type of problem. Asymptotically, the scalar field diverges and the potential vanishes. Hence, ω and $\omega_\phi\omega^3$ respectively diverges and vanishes for $\bar{\Omega} \rightarrow -\infty$. A theory with the same types of potential and behaviour at late times for the Universe has been studied in [85, 86]. However the coupling function was a constant and did not diverge at late times. Hence the corresponding Hamiltonian does not belong to the class we used in this work and will probably be oscillating at late times. Secondly, we examine $U = \phi^k$ with $k < 0$. Recently, this type of potential has been use to generate scaling solutions too [87]. Again the scalar field diverges asymptotically and the potential vanishes. At late time ω becomes a constant and $\omega_\phi\omega^{-3}$ vanishes. Thus, for these types of potentials, the scalar field is defined in $-\infty$ where it diverges and ω and U respect the solar system tests.

We conclude this discussion by summarising these results. We have shown that to obtain an isotropic expanding Universe at late times with a positive potential, we shall have $H > 0$, $\dot{H} < 0$ and $\bar{\Omega} \rightarrow -\infty$. This is necessary and sufficient for Einstein frame, sufficient and better for the Brans-Dicke frame since the gravitational constant does not diverge.

We have presented a method to obtain exact solution in the two frames. Then, considering the Einstein frame, we have recover Wald's theorem for Bianchi type I model.

The next results have been obtained by making the assumptions that the coupling function was such that the scalar field be defined in $\bar{\Omega} \rightarrow -\infty$ and that the Hamiltonian and thus the potential do not oscillate at late times.

Then, we have proved that when the 3-volume behaves asymptotically like an exponential, the Universe iso-

tropizes toward a De Sitter model and the potential became asymptotically a constant. Moreover, if at late times the scalar field is a constant different from zero, which seems to be a physically reasonable assumption if we consider measurements of the gravitational constant, the values of ω and $\omega_\phi \omega^{-3}$ are in agreement with the solar system tests. Reciprocally, when a non-oscillating potential becomes a constant asymptotically, the Universe tends toward a De Sitter model whatever ω in accordance with our assumptions. This generalizes Wald's result for the Bianchi type I model and shows that the De Sitter model is an attractor for this class of potential.
When the 3-volume behaves asymptotically as a power law of $\bar{t}$, the Universe isotropizes and the metric functions tend toward $\bar{t}^{2m}$. The potential will be positive if $m > 1/3$ and will vanish asymptotically. Thus, such a type of Universe solves the cosmological constant problem naturally. This enlightens the importance of power-law solutions in cosmology. If we assume that ϕ tends toward a constant, once more again, the coupling function respects the solar system tests. We can also express ω and $\omega_\phi \omega^{-3}$ asymptotically as some functions of ϕ, the potential and its derivative with respect to the scalar field. Then, we have shown that for an exponential potential $e^{k\phi}$ with $k > 0$, the coupling function and $\omega_\phi \omega^{-3}$ were in agreement with the solar system tests. ω can not be a constant since it diverges and thus, such theory will not tend toward a Brans-Dicke one. For a power law potential ϕ^k with $k < 0$, the coupling function tends toward a constant and $\omega_\phi \omega$ vanishes. So, the theory can tend toward Brans-Dicke theory and this constant have to be larger than 500 so that the theory respects the solar system tests at late times. For these two types of potentials, the scalar field diverges. We have checked that its asymptotic form was defined in $\bar{\Omega} \to -\infty$.

	Expansion in t_0	Isotropisation in t_0	$U_{BD} > 0$ in t_0
$\bar{\Omega}$ diverges	$\dot{\phi}/\phi > -2$, $H > 0$: $\bar{\Omega} \rightarrow -\infty$	Yes	$dH/dt > 0$ or $dH/d\bar{\Omega} < 0$
	$\dot{\phi}/\phi < -2$, $H < 0$: $\bar{\Omega} \rightarrow +\infty$	Yes if $\phi < e^{-6\bar{\Omega}}$	$dH/dt > 0$ or $dH/d\bar{\Omega} > 0$
$\bar{\Omega} \rightarrow cte$	$\dot{\phi}/\phi > -2$, $H > 0$	Yes if $\phi \rightarrow 0$	$dH/dt > 0$ or $dH/d\bar{\Omega} < 0$
	$\dot{\phi}/\phi < -2$, $H < 0$	Yes if $\phi \rightarrow 0$	$dH/dt > 0$ or $dH/d\bar{\Omega} > 0$

TAB. 6.2 – *Conditions for the Universe to be isotropic, in expansion and with a positive potential at late time in the Brans-Dicke frame.*

Chapitre 7

Occurence d'une singularité pour les modèles de Bianchi(1 article)

Un important problème en cosmologie est la présence de singularités, c'est-à-dire un ensemble de points de l'espace-temps où les lois classiques de la physique cessent d'être valable. On trouve de nombreux articles sur ce sujet dans la littérature. L'un des plus célèbre d'entre eux est celui d'Hawking et Penrose qui ont montré que, pour les modèles FLRW en présence de matière, il y avait toujours une singularité lorsque les conditions d'énergie fortes et faibles étaient respectées[88]. On peut également aborder ce problème du point de vue de la construction du Lagrangien d'une théorie de la gravitation comme l'a fait Brandenberger dans [89, 90, 91]. En cosmologie quantique l'absence de singularité est parfois imposée comme condition initiale en écrivant que la fonction d'onde de l'Univers est nulle lorsque les fonctions métriques le sont aussi. Enfin, dans le cadre de la théorie des cordes, Gasperini et Veneziano ont proposé le modèle de Pré-Big-Bang [92, 93] susceptible d'éviter la singularité. Dans tous les cas, il est toujours difficile de trouver des théories qui en sont dépourvues. Parmi les conditions nécessaires à leur absence, il faut que certains scalaires dont les scalaires de courbure, de Ricci et de Kretchmann ne divergent pas. Ce point de vue a été étudié quels que soient les invariants de courbure pour les modèles isotropes et la théorie tenseur-scalaire généralisée ($G = \phi^{-1}$) par S. K. Rama dans [94]. Dans ce travail nous établirons des conditions suffisantes permettant aux trois scalaires précités de rester finis dans le cadre de la théorie tenseur-scalaire hyperétendue définie par

$$L = G(\phi)^{-1}R - \frac{\omega(\phi)}{\phi}\phi_{,\mu}\phi^{,\mu}$$

et pour les modèles de Bianchi de type I, II, VI_0 et V. On espère ainsi obtenir des contraintes sur la forme que prendrait le Lagrangien d'une théorie de la gravitation d'où pourrait être absente une singularité grâce à la présence d'un champ scalaire. A cette fin, nous allons rechercher des conditions suffisantes portant sur les formes de $G(\phi)$ et $\omega(\phi)$ afin de pouvoir construire des théories précisées par la donnée de ces fonctions et telles que les invariants de courbure ne divergent pas. Mis à part le champ scalaire, nous ne considérerons pas d'autre type de contenu matériel comme des fluides parfaits. En effet, nous sommes intéressés par le comportement asymptotique de ces scalaires et les modèles vides de matière constituent souvent des limites asymptotiques pour les modèles avec fluide parfait. La relative simplicité mathématiques des équations de champs obtenues dans ce contexte nous permettra d'expliciter des conditions suffisantes à l'aide uniquement de G et ω afin d'éviter la divergence des scalaires que nous étudierons.

On pourrait objecter que cette étude est réalisée à un niveau classique alors que les singularités relèveraient plutôt d'une cosmologie quantique. Cependant il n'est pas déraisonnable de penser que l'absence de singularité pourrait être une prédiction réalisée par les théories de la gravitation tant au niveau classique que quantique.

Suffi cient conditions for curvature invariants to avoid divergencies in Hyperextended Scalar Tensor theory for Bianchi models

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Abstract

We look for sufficient conditions such that the scalar curvature, Ricci and Kretchmann scalars be bounded in Hyperextended Scalar Tensor theory for Bianchi models. We find classes of gravitation functions and Brans-Dicke coupling functions such that the theories thus defined avoid the singularity. We compare our results with these found by Rama in the framework of the Generalised Scalar Tensor theory for the FLRW models.

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7.1 Introduction

An important problem in cosmology is the presence of singularities, i.e. a set of points in spacetime where the laws of classical physics would be broken. This problem has been studied in numerous papers. One of the most famous is this of Hawking and Penrose [88]. It is shown that, for the FLRW models with matter field respecting the strong and weak energy conditions, it always exists a singularity. Some methods have also been developed to build Lagrangien such that the theory hence defined be non singular [89, 90, 91]. Another method is used in quantum cosmology where the absence of singularity is sometimes imposed by writing that the wave function vanishes with the scale factor. Last, for the string theory, Gasperini and Veneziano have proposed the models of Pre-Big-Bangs which could allow to avoid the singularity [92, 93].

In any case, to get a non singular theory it is necessary that all the curvature invariants be bounded. This point has been studied by Rama in [94] for the FLRW models and the Generalised scalar tensor theory (GST). In this paper we wish to examine from the same viewpoint what is the situation in the Hyperextended scalar tensor theory (HST) for the Bianchi models by studying the divergence of the scalar curvature, Ricci and Kretchmann scalars which are the most common curvature invariants met in the literature. Note that this type of study is made at a classical level whereas singularity deals with quantum cosmology. However, we hope that the absence of singularity at a classical level would indicate their absence at a quantum one.

Let us justify the geometrical framework of this paper. Although for present time our Universe seems to be isotropic, it is not proved that it was the case at early times or even that it is not a local phenomenon. Then, it is interesting to consider the homogenous models, i.e. the Bianchi models. Among them, the Bianchi types I, V, VII_0, VII_h and IX models, which admit FLRW solutions, are able to isotropize [95]. Hence we will study the Bianchi type I and V models. When they isotropize, the first one tends toward the flat isotropic model and the second one toward the open one. We will also study the Bianchi type VI_0 model, considered in [96, 97]. Last we will examined the Bianchi type II model which is representative of the Bianchi models of class A during phases of strong anisotropy [72].

Let us justify the study of the HST [35, 51]. Its Lagrangian contains two free functions depending on a scalar field ϕ. The first one, $G(\phi)$, represents the gravitational function and the second one, $\omega(\phi)$, a coupling function between the scalar field and the metric. The scalar fields are predicted by particle physics theories as string theory or supergravity. In cosmology they allow to solve numerous difficulties as age problem or inflationary exit. However, the use of theories with free functions depending on ϕ is also the source of new problems: what are the classes of functions G and ω which are agreed with both observational tests [64, 98, 99] and theoretical considerations such as the absence of singularity. In this work we will consider this last question: our goal is to find sufficient conditions on G and ω such that some curvature invariants do not diverge. We will not consider other forms of matter but scalar fields since scalar field dominated models are often asymptotical solutions for early or late times.

The paper is organised as follows. In section 7.2, we write the field equations of the HST, the scalar curvature, Ricci and Kretchmann scalars. In section 7.3, we determine sufficient conditions such that they be bounded at any times for the Bianchi type I, II, V and VI_0 models. In section 7.4, we use them to determine some suitable forms of ω for the GST and a string inspired theory. We conclude in section 7.5 and compare our results with these of Rama.

7.2 The curvature invariants

We use the following line element:

$$ds^2 = -dt^2 + e^{2\alpha}(\omega^1)^2 + e^{2\beta}(\omega^2)^2 + e^{2\gamma}(\omega^3)^2 \tag{7.1}$$

The ω^i are the one forms specifying each Bianchi model. The Lagrangien of the HST is written:

$$L = G(\phi)^{-1}R - \frac{\omega(\phi)}{\phi}\phi_{,\mu}\phi^{,\mu} \tag{7.2}$$

where ϕ is the scalar field, ω the coupling function and G the gravitation function, both depending on ϕ. We get the field equations and the Klein-Gordon equation by varying the action with respect to the metric functions and the scalar field:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \quad = \quad G[\frac{\omega}{\phi}\phi_{,\mu}\phi_{,\nu} - \frac{\omega}{2\phi}\phi_{,\lambda}\phi^{\lambda}g_{\mu\nu} + (G^{-1})_{,\mu;\nu} - g_{\mu\nu}\Box(G^{-1})] \tag{7.3}$$

$$\dot{\phi}^2\left[-\frac{\omega_\phi}{\phi} + \frac{\omega}{\phi^2} - G(G^{-1})_\phi\frac{\omega}{\phi}\right] + \frac{2\omega}{\phi}\Box\phi + 3G(G^{-1})_\phi\Box G^{-1} = 0 \tag{7.4}$$

An overdot means a derivative with respect to the proper time t. To calculate the curvature invariants, we define the τ time by $dt = e^{\alpha+\beta+\gamma}d\tau$. It would be more interesting to use the proper time t, since τ is not a physically significant times. However calculus in the Bianchi model are more tractable in the τ time. In fact, we will first get our results in the τ time and will generalise then in the t times in the last section by making comparisons with the results of Rama for the FLRW models.

The first curvature invariant we compute is the scalar curvature, obtained by contracting the equation (7.3):

$$R = V^{-2}G(-\omega\phi^{-1}\phi'^2 - 3(G^{-1})'') \tag{7.5}$$

The prime holds for derivative with respect to τ and $V = e^{\alpha+\beta+\gamma}$ defines the 3-volume of the Universe. We introduce (7.5) in (7.3) to obtain an expression for $R_{\mu\nu}$ and then we get the Ricci scalar:

$$\begin{aligned} R_{\mu\nu}R^{\mu\nu} \quad = \quad & V^{-4}G^2[\omega^2\phi^{-2}\phi'^4 + \omega\phi^{-1}\phi'^2(3(G^{-1})'' - 2(G^{-1})'V'V^{-1}) \\ & +(-(G^{-1})''^2 + (G^{-1})'^2V^{-2}V'^2 - 2(G^{-1})''(G^{-1})'V^{-1}V')] \end{aligned} \tag{7.6}$$

Last, the Kretchmann scalar defined as $R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}$ will be calculated with the help of:

$$R_{\alpha\beta\mu\nu} = \Gamma_{\alpha\beta\nu,\mu} - \Gamma_{\alpha\beta\mu,\nu} + \Gamma^m_{\beta\nu}\Gamma_{\alpha m\mu} - \Gamma^m_{\beta\mu}\Gamma_{\alpha m\nu} - C^m_{\mu\nu}\Gamma_{\alpha\beta m} \tag{7.7}$$

with

$$\Gamma_{\alpha\beta\mu} = 1/2(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} + C_{\mu\alpha\beta} + C_{\beta\alpha\mu} - C_{\alpha\beta\gamma}) \tag{7.8}$$

The Γ are the connections and the C the structure constants specifying each Bianchi model. To express this scalar as a function of V, ω, G and ϕ, as the two previous ones, we need to solve the field equations (7.3)-(7.4) to get α, β and γ depending on this quantities. In the next section we choose sufficient conditions such that the three curvature invariants be bounded.

7.3 Sufficient conditions such that the scalar curvature, Ricci and Kretchmann scalars be bounded

The Klein-Gordon equation can be integrated to give:

$$\left[\frac{3}{4}(G^{-1})^2_\phi + \frac{1}{2\phi}G^{-1}\omega\right]\phi'^2 = \phi_0 \tag{7.9}$$

ϕ_0 is an integration constant. The scalar field is thus a monotonous function of time. The term in square bracket is proportional to the energy density of the scalar field in the Einstein frame. If we assume a positive energy density, we get a variation interval for ϕ. Moreover, equation (7.9) allows us to write ϕ', $(G^{-1})' = G_\phi^{-1}\phi'$ and $(G^{-1})'' = ((G^{-1})')_\phi\phi'$ as functions of ω and G:

$$(G^{-1})' = \phi_0^{1/2}(G^{-1})_\phi(\frac{3}{4}(G_\phi^{-1})^2 + \frac{G^{-1}\omega}{2\phi})^{1/2} \tag{7.10}$$

$$(G^{-1})'' = 4\phi_0\frac{2(G^{-1})_{\phi\phi}\omega G^{-1}\phi - (G^{-1})_\phi^2\omega\phi + (G^{-1})_\phi(\omega G^{-1} - G^{-1}\omega_\phi\phi)}{(2G^{-1}\omega + 3\phi(G^{-1})_\phi^2)^2} \tag{7.11}$$

Our aim being to choose sufficient conditions on G and ω such that the curvature invariants do not diverge, we have just to write them as function of G, ω, ϕ and their derivatives with respect to τ and to use the expressions (7.10) and (7.11) to achieve our goal. In the first subsection, we look for these sufficient conditions. Sometimes, their expressions depend on the Bianchi type. They will be studied in the second subsection.

7.3.1 Sufficient conditions such that the curvature invariants be bounded

Each of the three curvature invariants depends on the 3-volume. So the first sufficient condition we will choose will be $V \neq 0$. Its expression as a function of the scalar field depends on the Bianchi model and will be studied in the next subsection. Assuming that $V \neq 0$, sufficient conditions such that the scalar curvature (7.5) be bounded whatever the Bianchi type will be that the following quantities do not diverge:

- $G(G^{-1})''$
- $G\omega\phi'^2\phi^{-1}$

Each of them may be expressed as a function of G and ω independently of the Bianchi type.
For the Ricci scalar, it is sufficient that the following quantities do not diverge:

- $G(G^{-1})''$
- $G\omega\phi'^2\phi^{-1}$
- $G(G^{-1})'$
- $V'V^{-1}$ i.e. $\alpha' + \beta' + \gamma'$

The expression of the last one as a function of G and ω depends on the Bianchi model and will be studied in the next subsection. The two first conditions have already been chosen for the scalar curvature. The third one is new. Its expression as a function of G and ω does not depend on the Bianchi type and can be written with help of (7.10) and (7.11).
The expressions of all the sufficient conditions as function of G and ω such that the Kretchmann scalar be bounded depends on the Bianchi model. For the Bianchi type I and V model, it is sufficient that the first and second derivatives of α, β and γ be bounded. For the Bianchi type II model, we have an additional conditions, i.e. α have to be bounded. Idem for the Bianchi type VI_0 model for which α and β have not to diverge. These conditions always imply that $V'V^{-1}$ is bounded. Of course, requiring that the derivatives of α, β and γ do not diverge for the Kretchmann scalar is also sufficient such that the two previous curvature invariants be bounded. Then, the sufficient conditions we found above as function of G and ω are contained in the requirement that these derivatives be bounded. However, they have been found independently of any Bianchi model. It is why we have considered that it was interesting to deduce them separately.

7.3.2 Expression of the sufficient conditions depending on the Bianchi model as function of the scalar field

In what follows, we examine the previous conditions whose expressions as function of the scalar field depends on the Bianchi model, i.e. $V = e^{\alpha+\beta+\gamma} \neq 0$ and the conditions related to the Kretchmann scalar. The structure constants of the Bianchi type I model are all vanishing. The spatial components of the field equations are:

$$\begin{aligned} \alpha'' &= -\alpha' G(G^{-1})' - \tfrac{1}{2}G(G^{-1})'' \\ \beta'' &= -\beta' G(G^{-1})' - \tfrac{1}{2}G(G^{-1})'' \\ \gamma'' &= -\gamma' G(G^{-1})' - \tfrac{1}{2}G(G^{-1})'' \end{aligned} \tag{7.12}$$

We multiply each of them by G^{-1}. After an integration, we get:

$$\alpha' = (K - 1/2(G^{-1})')G \tag{7.13}$$

From a second integration, we deduce:

$$\alpha = K \int G\phi'^{-1} d\phi - 1/2 \ln G^{-1} \tag{7.14}$$

K is an integration constant. Equivalent expressions can be found for β and γ. With simple considerations, we get some sufficient conditions such that the derivatives of α, β and γ be finite and $V \neq 0$:
- $K \int G\phi'^{-1} d\phi$ does not diverge toward $-\infty$
- G is bounded and non vanishing
- $G(G^{-1})'$ is bounded
- $G(G^{-1})''$ is bounded

Each of them can then be written as function of the scalar field by using equations (7.10) and (7.11). We will study their physical meaning in the last section. In [29] where a GST with a perfect fluid in the Bianchi type I model is considered, similar conditions for the absence of singularity was found: it has been shown that singularity occurs when $V \rightarrow 0$ and $\phi \rightarrow \infty$.

Sufficient conditions such that e^α and the first and second derivatives of α, β and γ be bounded for the Bianchi type II model and the metric functions be non vanishing.

The non vanishing structure constants are $C^1_{23} = -C^1_{32} = 1$. The spatial components of the field equations are written:

$$\begin{aligned} \alpha'' &= -\alpha' G(G^{-1})' - \tfrac{1}{2} G(G^{-1})'' - 1/2 e^{4\alpha} \\ \beta'' &= -\beta' G(G^{-1})' - \tfrac{1}{2} G(G^{-1})'' + 1/2 e^{4\alpha} \\ \gamma'' &= -\gamma' G(G^{-1})' - \tfrac{1}{2} G(G^{-1})'' + 1/2 e^{4\alpha} \end{aligned} \tag{7.15}$$

In the Einstein frame where the metric functions are related to these of the Brans-Dicke frame by $g_{\mu\nu} = G\tilde{g}_{\mu\nu}$, the solutions of the field equations are well known. They are written $\tilde{\alpha} = 1/2 \ln(k \cosh^{-1}[\tilde{\tau} - \tilde{\tau}_0])$, $\tilde{\beta} = B_0 + B_1 \tilde{\tau} - 1/2 \ln(k \cosh^{-1}[\tilde{\tau} - \tilde{\tau}_0])$ and a similar expression for γ. k, B_0, B_1 and $\tilde{\tau}_0$ are integration constants. From the Klein-Gordon equation in the Einstein frame, we get $\tilde{\tau} - \tilde{\tau}_0 = \int G/\phi' d\phi$. Hence, we deduce the expression of the metric functions in the Brans-Dicke frame:

$$\begin{aligned} \alpha &= 1/2 \ln\{kG \cosh^{-1}(k \textstyle\int G/\phi' d\phi)\} \\ \beta &= B_0 + B_1 \textstyle\int G\phi'^{-1} d\phi - 1/2 \ln\{kG^{-1} \cosh^{-1}(k \textstyle\int G\phi'^{-1} d\phi) \\ \gamma &= C_0 + C_1 \textstyle\int G\phi'^{-1} d\phi - 1/2 \ln\{kG^{-1} \cosh^{-1}(k \textstyle\int G\phi'^{-1} d\phi) \end{aligned}$$

Thus, some sufficient conditions such that e^α and the first and second derivatives of α, β and γ be bounded for the Bianchi type II model with $V \neq 0$ will be:
- $\int G\phi'^{-1} d\phi$ is bounded.
- G is bounded and non vanishing.
- $G(G^{-1})'$ is bounded.
- $G(G^{-1})''$ is bounded.

These conditions are the same as these of the Bianchi type I model but now $\int G\phi'^{-1} d\phi$ have to be bounded such that e^α stays finite.

Sufficient conditions such that e^α, e^β, the first and second derivatives of α, β and γ be bounded for the Bianchi type VI_0 model and the metric functions be non vanishing.

The non vanishing structure constants are $C^1_{23} = -C^1_{32} = C^2_{13} = -C^2_{31} = 1$. We will consider the LRS case for which $\alpha = \beta$. The spatial components of the field equations are written:

$$\begin{aligned} \alpha'' &= -\alpha' G(G^{-1})' - \tfrac{1}{2} G(G^{-1})'' \\ \gamma'' &= -\gamma' G(G^{-1})' - \tfrac{1}{2} G(G^{-1})'' + 2e^{4\alpha} \end{aligned} \tag{7.16}$$

The first equation is the same as for the Bianchi type I model. Its solution is then:

$$\alpha = K \int G/\phi' d\phi - 1/2 \ln G^{-1}$$

Putting it in the second equation of (7.16), we get:

$$\gamma = -1/2 \ln G^{-1} + (\int G/\phi' d\phi)^2 + 2\int e^{4K \int G/\phi' d\phi}/\phi' d\phi$$

Some sufficient conditions such that e^{α}, e^{γ}, the first and second derivatives of α and γ be bounded with $V \neq 0$ are thus:

- $\int G\phi'^{-1} d\phi$ is bounded.
- G is bounded and non vanishing.
- $G(G^{-1})'$ is bounded.
- $G(G^{-1})''$ is bounded.
- $\int e^{4K \int G\phi'^{-1} d\phi} \phi'^{-1} d\phi$ does not tend toward $-\infty$.

These conditions are the same as these of the Bianchi type II model except the last one which seems to be specific to the Bianchi type VI_0 model.

Sufficient conditions such that the first and second derivatives of α, β and γ be bounded for the Bianchi type V model and the metric functions be non vanishing.

The non vanishing structure constants of this model are $C^2_{21} = -C^2_{12} = C^3_{31} = -C^3_{13} = 1$. The spatial components of the field equations are written:

$$\begin{aligned} \alpha'' &= -\alpha' G(G^{-1})' - \frac{1}{2} G(G^{-1})'' + 2e^{2\beta+2\gamma} \\ \beta'' &= -\beta' G(G^{-1})' - \frac{1}{2} G(G^{-1})'' + 2e^{2\beta+2\gamma} \\ \gamma'' &= -\gamma' G(G^{-1})' - \frac{1}{2} G(G^{-1})'' + 2e^{2\beta+2\gamma} \end{aligned} \tag{7.17}$$

In the Einstein frame, these three equations are turned into General Relativity equations for the Bianchi type V model whose solutions in the $\tilde{T}$ time defined by $d\tilde{\tau} = d\tilde{T} e^{-\tilde{\beta}-\tilde{\gamma}} = G d\tau$ have been found by Joseph [100]:

$$\begin{aligned} e^{2\tilde{\alpha}} &= K^2 \sinh(2\tilde{T}) \\ e^{2\tilde{\beta}} &= K^2 \sinh(2\tilde{T}) \tanh(\tilde{T})^{\sqrt{3}} \\ e^{2\tilde{\gamma}} &= K^2 \sinh(2\tilde{T}) \tanh(\tilde{T})^{-\sqrt{3}} \end{aligned}$$

We then calculate that:

$$\tilde{\tau} - \tilde{\tau}_0 = 1/2 K^{-2} \ln \tanh \tilde{T} = \int G\phi'^{-1} d\phi \tag{7.18}$$

The central member of this last expression is defined for $\tilde{T} \in [0, +\infty[$ and vary from $-\infty$ to 0. We deduce that the integral diverges when $\tilde{T} \to 0$ and vanishes when $\tilde{T} \to +\infty$. So, some sufficient conditions such that the first and second derivatives of α, β and γ be bounded with $V \neq 0$ are:

- $\tilde{T}$ is bounded and non vanishing, i.e. $\int G\phi'^{-1} d\phi$ do not diverge toward $-\infty$ and is non vanishing.
- G is bounded and non vanishing.
- $G(G^{-1})'$ is bounded.
- $G(G^{-1})''$ is bounded.

These conditions are similar to these of the Bianchi type I model but the integral of G with respect to τ shall be non vanishing. This is in agreement with [29] where it was noticed that the behaviour of the Bianchi type V model near the singularity is a subset of the Bianchi type I model.

A summarise of these results is presented on tables 7.1 and 7.2.

7.4 Applications

In this section, we use the sufficient conditions of tables 7.1 and 7.2 to find some forms of the functions G and ω such that the scalar curvature, the Ricci and Kretchmann scalars do not diverge for GST and string inspired theories.

7.4.1 Generalised scalar tensor theory

The GST is defined by $G^{-1} = \phi$. Lots of papers are devoted to its study [58, 54, 52, 29]. For this class of theories, we have calculated that:

$$\begin{aligned} G &= 1/\phi \\ G(G^{-1})' &\propto \phi^{-1}(3+2\omega)^{-1/2} \\ G(G^{-1})'' &\propto -\omega_\phi\phi^{-1}(3+2\omega)^{-2} \\ G\omega\phi'^2\phi^{-1} &\propto \omega\phi^{-2}(3+2\omega)^{-1} \end{aligned}$$

From table 7.2 we see that whatever the Bianchi type, G have to be bounded and non vanishing. Hence, ϕ is strictly positive or negative and bounded. Since $G(G^{-1})'$ have not to diverge, we shall ask also that $3+2\omega$ be non vanishing. This last function have to be positive such that the energy density of the scalar field be positive in the Einstein frame. Last, $G(G^{-1})''$ have also to be bounded: from what we write for ϕ, we deduce that it will be verified if it is also the case of $\omega_\phi\omega^{-2}$. All these conditions imply that $G\omega\phi'^2\phi^{-1}$ will stay bounded.
A function ω corresponding to these requirements will be, as instance, $3+2\omega = m + [-(i+\phi)(j+\phi)]^n$ with $m > 0$, $(i,j) < (0,0)$ and $n > 1$. The scalar field is then defined on the closed interval $[-i, -j]$. As $\phi' \propto 1/\sqrt{3+2\omega}$ and G does not diverge, we can show that the integrals of the table 7.2 stay finite. Numerically, we have checked that $\int G\phi'^{-1}d\phi$ is non vanishing. Hence, for all the Bianchi models we have studied, none of the three curvature invariants diverges.
The low energy action of the string theory without antisymetric strength field is a HST with $G^{-1} = \omega = e^{-\phi}$. In this application, we will choose $G^{-1} = e^{-\phi}$ and $\omega = e^{-\phi}\Omega(\phi)$. At early time, the compactification of extra dimensions could give birth to physical phenomenon which would be described by such theories [71, 72]. It is then interesting to find these which are non singular. We have:

$$\begin{aligned} G &= e^\phi \\ G(G^{-1})' &\propto -2\phi_0 e^\phi(3+2\phi^{-1}\Omega)^{-1/2} \\ G(G^{-1})'' &\propto 4\phi_0^2 e^\phi(-\Omega+\phi\Omega_\phi)(3\phi+2\Omega)^{-2} \\ G\omega\phi'^2\phi^{-1} &\propto e^{2\phi}(3\phi\Omega^{-1}+2)^{-1} \end{aligned}$$

G will be bounded and non vanishing if ϕ is bounded. From this, we deduce that $G(G^{-1})'$ is bounded and real if $\Omega\phi^{-1} > -3/2$. Then, $G(G^{-1})''$ is bounded if $\phi^2\Omega_\phi\Omega^{-2}$ is finite. All these conditions imply that $G\omega\phi'^2\phi^{-1}$ is always finite. A function Ω corresponding to these requirements, have the same form as in the previous subsection with $(i,j) < (0,0)$, $n > 1$, $m > -3/2$ and $mi^{-1} < 3/2$. Then, the same remarks as in subsection 7.4.1 apply here.

7.5 Final remarks and conclusion

In this work, we have determined sufficient conditions such that the scalar curvature, the Ricci and Kretchmann scalars do not diverge for the HST in Bianchi models. It is necessary such that the theory be non singular. These conditions are summarised in table 7.1 and 7.2.
Of course, other types of sufficient conditions can be chosen from this work. As instance, we can replace the two conditions "$G(G^{-1})'$ is bounded" and "$V \neq 0$" by "$G(G^{-1})'V^{-1} = G\dot{G}^{-1}$ is bounded". This new condition can be written as a function of ϕ by help of (7.10) and the expressions of $V(\phi)$ for each Bianchi model. It is even possible to write the three curvature invariants as functions of ϕ, G, ω and their derivatives with respect to ϕ and to search conditions such that the invariants be bounded directly from these expressions. However, it is not an easy task, particularly when we consider the Kretchmann scalar.
What are the physical interpretation of the conditions we have chosen? Whatever the Bianchi models, the gravitation function G have to be bounded and non vanishing. It follows that $(G^{-1})'$ and $(G^{-1})''$ are bounded. Another conditions is that $\int G\phi'^{-1}d\phi = \int Gd\tau$ is bounded. Hence, we deduce that the sign of G will not change during time evolution: the gravitation is either attractive or repulsive but can not change its nature. If G tends toward a constant as it seems to be the case for our present epoch or is asymptotically monotone, as $\int Gd\tau$ is bounded, τ is bounded. As $dt = Vd\tau$, it means that if the 3-volume of the Universe diverge or tends asymptotically toward a constant, t will behave in the same way. Hence, a finite asymptotic value of the Universe 3-volume means a finite interval of proper time and an infinite asymptotical value, an

open interval of proper time for the Universe. We have also chosen that $G\omega\phi'^2\phi^{-1}$ be bounded. As G do not diverge, if we impose that the solar system tests be respected, i.e. $\omega \to \infty$, we need then $\phi'\phi^{-1} \to 0$. As instance, It could be realised if the scalar field tended toward a constant which is a realistic assumption for late times period. Note that the condition on $G\omega\phi'^2\phi^{-1}$ is not independent of the others: it is a consequence of the Klein-Gordon equation, the fact that G^{-1} is non vanishing, bounded and $(G^{-1})'$ is bounded. This fact has been observed in the applications of section 7.4.
Finally, the conditions we have established such that the three curvature invariants we studied be finite can be summarised in four simple points for the Bianchi types I, II, V and VI models:

- G is bounded and non vanishing
- $(G^{-1})'$ and $(G^{-1})''$ which can be expressed with equations (7.10) and (7.11) are bounded
- $\int G d\tau$ does not diverge toward $-\infty$ or/and $+\infty$ depending on the Bianchi models
- For the Bianchi type VI_0 model there is a special conditions, i.e. $\int e^{4K\int G d\tau} d\tau$ does not diverge.

From these conditions and the fields equations obtained for each Bianchi model, we deduce that the n^{th} derivatives of each metric function with respect to τ will be bounded if the n^{th} derivative of G with respect to τ is bounded. This derivative can be calculated as a function of G, ω and their derivatives with respect to the scalar field with help of the recursive relation $d^nG/d\tau^n = d\left[d^{n-1}G/d\tau^{n-1}\right]/d\phi\phi'$ and the relation (7.9). Adding this last conditions to the four previous one, this ensures then that each curvature invariant is bounded.
Each of these conditions can be expressed with G, ω and their derivatives with respect to the scalar field. Hence we have achieved the goal we fixed at the beginning of the paper: we have found a simple set of sufficient conditions such that the invariant curvatures of the HST be bounded. The theories defined by these conditions could then be interesting theories to represent asymptotical behaviour of an anisotropic Universe if we assume that the singularity must be avoided.

A similar work has been carried out by Rama [94], with the GST with a perfect fluid for the FLRW models. In this last paper, as sufficient conditions such that none of the curvature invariant diverges, it was chosen that the invert of the scale factor, e^A, and the successive derivatives of A with respect to the proper time t be bounded. If we exclude the conditions specific to the presence of matter, the others were written:

- e^{-A} is bounded
- $\dot\phi\phi^{-1}$ is bounded
- $\omega\dot\phi^2\phi^2$ is bounded
- $\phi^n(3+2\omega)^{-1}d^n(3+2\omega)/d\phi^n(\dot\phi\phi^{-1})^n$ is bounded

We have recovered the first one by writing that the 3-volume be non vanishing. It implies also that $\int G d\tau$ is bounded. Since we have chosen that $G(G^{-1})'$ and V^{-1} are bounded, it means that there product is bounded too. This implies the second condition since $dt = V d\tau$ and ϕ should be replaced by G^{-1} for the hyperextended theory. However the reverse is false. In this way, the sufficient conditions we have chosen are more restrictive for the functions G and ω than these of Rama. We can recover the third condition of Rama in the same way since we have assumed that V^{-1} and $G\omega\phi'^2\phi^{-1}$ were bounded. the fourth condition come from the fact that the successive derivatives of A have to be bounded and should be related with the condition on the finiteness of $d^nG/d\tau^n$. Hence, by matching the conditions chosen in this work, we can recover all the conditions of [94] which are not concerned by the presence of matter. The main difference comes from the fact that we have replaced ϕ by G^{-1}, i.e it arises because we have considered an hyperextended rather than a generalised scalar tensor theory. It is a difference of physical order. There is also another difference which is of geometrical order. The only condition present in this work and not in [94] is the one for the Bianchi type VI_0 model, implying that $\int e^{4K\int G\phi'^{-1}d\phi}\phi'^{-1}d\phi$ is bounded. It seems to be a specific condition characterising this model since it has no equivalent in the other Bianchi models. It would explain why it does not appear in the paper of Rama since Bianchi type VI_0 model can not be related to a FLRW one. Hence, this paper complete [94] by extended some of its results in the HST for Bianchi models which was one of the issues evoked in its conclusion.

curvature invariant	Bounded quantities
R	$G(G^{-1})''$, $G\omega\phi'^2\phi^{-1}$, α', β', γ'
$R_{\mu\nu}R^{\mu\nu}$	$G(G^{-1})''$, $G\omega\phi'^2\phi^{-1}$, $G(G^{-1})'$, α', β', γ'
$R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$	
I et V	α', β', γ', α'', β'', γ''
II	α', β', γ', α'', β'', γ'', e^α
VI_0	α', β', γ', α'', β'', γ'', e^α,e^γ

TAB. 7.1 – *When $V \neq 0$, it is sufficient that these quantities be bounded such that the curvature invariants do not diverge. For the scalar curvature and the Ricci scalar, these conditions are independent on the considered Bianchi models.*

Model	Conditions
I	G, $G(G^{-1})'$, $G(G^{-1})''$ are bounded, G is non vanishing
	$K\int G\phi'^{-1}d\phi$ does not diverge toward $-\infty$
II	G, $G(G^{-1})'$, $G(G^{-1})''$ is bounded, G is non vanishing
	$\int G\phi'^{-1}d\phi$ is bounded
VI_0	G, $G(G^{-1})'$, $G(G^{-1})''$ is bounded, G is non vanishing
LRS	$\int G\phi'^{-1}d\phi$ is bounded
	$\int e^{4K\int G\phi'^{-1}d\phi}\phi'^{-1}d\phi$ does not tend toward $-\infty$
V	G, $G(G^{-1})'$, $G(G^{-1})''$ is bounded, G is non vanishing
	$\int G\phi'^{-1}d\phi$ is non vanishing and does not diverge toward $-\infty$

TAB. 7.2 – *Sufficient conditions such that the Kretchmann scalar be bounded and the 3-volume be non vanishing for the Bianchi type I, II, VI_0 and V models.*

Chapitre 8

Symétries de Noether des modèles FLRW(1 article)

Dans ce chapitre, nous étudions les modèles homogènes d'un point de vue radicalement différent de celui des chapitres précédents. En effet, nous nous placerons dans le cadre des cosmologies homogènes et isotropes FLRW et surtout nous nous intéresserons à la présence de symétries de Noether. La théorie que nous considérerons est la théorie tenseur-scalaire hyperétendue (HST) définie par le Lagrangien

$$L = \left[G(\phi)^{-1}R + \omega\phi^{-1}\phi_{,\mu}\phi^{,\mu} - U\right]\sqrt{-g} + L_m$$

Il comporte une fonction de gravitation G, une constante de couplage ω et un potentiel U, chacune de ces fonctions étant dépendante du champ scalaire ϕ, et L_m représente le Lagrangien d'un fluide parfait. C'est donc la théorie la plus complète que nous ayons étudiée jusqu'à présent et un moyen radical de contraindre les formes de G, ω et U est, comme nous allons le voir, d'exiger qu'elle soit compatible avec les symétries de Noether.

Bien sûr rien ne prouve qu'une telle caractéristique soit nécessaire à une théorie de la gravitation du type tenseur-scalaire. Expliquons l'intérêt de ces symétries. Le théorème des symétries de Noether établit que pour chaque symétrie continue des lois de la physique, il doit exister une loi de conservation et réciproquement, pour chaque loi de conservation, il doit exister une symétrie continue. Dans ce travail, ces symétries seront étudiées via l'approche de Ritis et al [101] et Capozziello et al [102]. Nous considérerons un Lagrangien L et un champ de vecteurs χ et nous rechercherons la symétrie définie par la dérivée de Lie $\ell_\chi L$. Ceci nous permettra de trouver une relation devant être respectée entre les trois fonctions G, ω et U afin qu'une symétrie de Noether existe. Il est alors possible de déduire les quantités conservées et des solutions exactes mais nous n'irons pas jusque là. Ce dernier type de calculs doit d'ailleurs être effectué avec prudence: il a été récemment montré que ces symétries, dans certains cas, ne sont pas consistantes avec les équations de champs et que d'autres types de symétries peuvent exister[103].

Le contexte géométrique de ce travail sera les modèles FLRW. Il constitue une généralisation importante des travaux effectués dans [101] et dans [102]. Il serait intéressant d'étendre cette étude aux modèles de Bianchi mais les équations deviennent rapidement inextricables dès le modèle de Bianchi de type I et une adaptation de la méthode utilisée ici est nécessaire et reste à définir.

L'avantage de cette méthode est donc d'imposer des contraintes très fortes sur les théories tenseur-scalaires mais l'inconvénient, c'est de supposer la présence d'une symétrie de Noether, ce qui n'est pas toujours aisément justifiable.

Noether Symmetry of the Hyperextended Scalar Tensor theory for the FLRW models

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Abstract

We study in which conditions the Hyperextended Scalar Tensor theory in an FLRW background admits a Noether symmetry and derive the vectors field generating it.

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8.1 Introduction

One of the most well-known scalar tensor theories is the Brans-Dicke one [7] developed in the sixties. This interest was motivated by the fact that it is able to reconcile a relativistic theory of gravitation with the ideas of Mach. It introduced the ideas of a gravitational coupling function, G, varying as the inverse of the scalar field ϕ and thus depending on the time[6] and of a coupling constant, ω, between the scalar field and the metric functions. With the discovery of inflation by Guth [8] in the eighties allowing explaining why the Universe could be so flat or so isotropic, the interest for the scalar tensor theories has increased and found new justifications. Inflation could have been recently detected with the supernovae of type IA[9, 10] and is often interpreted as the presence of a cosmological constant Λ in the field equations. In the same time, physical particle progress have shown the importance of massive scalar field or varying gravitational functions. As instance the gravitational and the Brans-Dicke coupling functions of the low energy effective string action are exponential laws of ϕ. Moreover, unification theories predict a cosmological constant larger than the presently observed value. One way to solve this cosmological constant problem could be to consider a varying potential instead of a constant one. Thus the connection between particle physics and cosmology encourage us to consider scalar-tensor theories more general than the Brans-Dicke one. The Lagrangian of the Hyperextended Scalar Tensor theories (HST) seems to be suited to take into account this need for generality since it is written with a gravitational function $G(\phi)$ and a Brans-Dicke coupling function $\omega(\phi)$. It is why we have chosen to study it when a potential $U(\phi)$ and a perfect fluid are presents.

The HST thus defined has then 3 undetermined functions. This is an advantage and a drawback at the same time. The advantage comes from the fact that any result we will obtain from this theory will be very general and could be applied to a large number of scalar tensor theories simply by assuming some special forms for G and ω. As instance, General Relativity with a scalar field is obtained with $G = G_0$ and Brans-Dicke theory for $G^{-1} = \phi$ and $\omega = \omega_0$. The drawback is that there are few indices indicating us what should be the physically interesting forms of the three undetermined functions depending on the scalar field. We can try to determine some of their characteristics from an observational point of view. Hence, in [41] it is shown how from the observations, it could be possible to determine the full Lagrangian and thus the potential from the luminosity distance and the linear density perturbation in the dust like matter as function of redshift. In [64], the convergence toward General Relativity, the presence of singularity or the dynamical evolution of the Universe at any time have been studied depending on the form of ω. In [104], observation of the variation of the fine-structure constant is analysed, giving us restriction on the possible variation of G. We can also leave the cosmological principle, assuming an anisotropic Universe and looking for the forms of the functions allowing the isotropy[105]. Considering the relations of this theory with the particle physics, another possibility is to claim that the HST could respect some of its symmetries as Noether symmetries. This is the approach chosen in [106, 101] and that we will follow in this work. Our goal will be to look for the existence conditions of a Noether symmetry for the HST in different physical (in the vacuum (i.e. with a non-massive scalar field), with a potential or with a perfect fluid) and geometrical (flat open and closed Universe) contexts. Lets note that a transformation of the scalar field, $G^{-1} = \Phi$ allows to reduce the number of the undetermined function from three to two. The theory thus defined is named Generalised Scalar Tensor theory (GST) and has been studied from the same point of view in [106, 101]. However, these results can not be extended to HST by help of an inverse transformation and thus be related on important theories as the effective low energy string theory. The two classes of theories are physically equivalent but it is not possible to know the constraint imposed by the Noether symmetry on G, U and ω from these determined for the GST.

Lets explain the interest of the Noether symmetries [1]. Noether symmetries theorem states that for every continuous symmetry of the laws of physics, there must exist a conservation law and reciprocally for every conservation law, there must exist a continuous symmetry. In this work, it will be studied via the approach of de Ritis et al [101] and Capozziello et al [102]. We will consider a point Lagrangian L and a vector field χ. A first step is to find the symmetry χ, defined by the Lie derivative $\ell_\chi L$. The second step that we will not consider in this paper, is to determine the conserved quantity Q that can be found by computing the Cartan one form $\theta_L = \frac{\partial L}{\partial \dot{a}} da + \frac{\partial L}{\partial \dot{\phi}} d\phi$, contracting it with χ to get $Q = i_\chi \theta_L$. Then, the calculus of Q can allow to solve exactly the field equations as shown in the previously quoted papers. Thus Noether symmetry is very important in the search for exact solutions of theories having particular symmetries and conserved quantities, helping the study of more general ones. However, note that recently, it has been demonstrated that Noether symmetry could, for some cases, not be consistent with dynamical equations and that other type of symmetries could exist[103] indicating that in the future we will have to proceed with care when we consider the second step.

The geometrical framework of this study will be the FLRW models. It would be more logical to consider an anisotropic and inhomogeneous model more qualified to describe the geometry of the early Universe where particle physics naturally takes place. However, it does not exist a full classification of these geometries contrary to the FLRW or Bianchi models. The relative simplicity of the FLRW models will allow us to study the Noether symmetries for a whole class of geometrical models and thus we will be able to make comparisons between each of them.

The plane of this paper is the following: in the section 8.2, we look for the conditions allowing a Noether symmetry for the HST in the FLRW models. We discuss our results and conclude in the section 8.3.

8.2 Noether symmetry of the FLRW

We use the following form of the metric describing an isotropic and homogeneous Universe:

$$ds^2 = -dt^2 + a^2 d\Omega \tag{8.1}$$

a being the scale factor. The Lagrangian of the HST with a potential and a perfect fluid is written:

$$L = \left[G(\phi)^{-1} R + \omega \phi^{-1} \phi_{,\mu} \phi^{,\mu} - U\right] \sqrt{-g} + L_m \tag{8.2}$$

G being the gravitational coupling function, ω the Brans-Dicke coupling function, U the potential, ϕ the scalar field from which depends on previous quantities and L_m the Lagrangian corresponding to a perfect fluid with equation of state $p = (\delta - 1)\rho$. Using the fact that $\int \Box G^{-1}\sqrt{-g} = 0$, the point Lagrangian for the FLRW models is written:

$$L = -6G^{-1} a \dot{a}^2 - 6G_\phi^{-1} a^2 \dot{a}\dot{\phi} + \omega a^3 \phi^{-1} \dot{\phi}^2 + 6kaG^{-1} - a^3 U + \rho_0(\gamma - 1) a^{3(1-\gamma)} \tag{8.3}$$

To find the conditions for Noether symmetry, we will follow the approach of de Ritis et al [101] and Capozziello et al [102]. We will consider the configuration space $E = (a,\phi)$ whose corresponding tangent space is $TE = (a,\dot{a},\phi,\dot{\phi})$. The vector field X generating the symmetry is then:

$$X = \alpha \frac{\partial}{\partial a} + \chi \frac{\partial}{\partial \phi} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\chi} \frac{\partial}{\partial \dot{\phi}} \tag{8.4}$$

where α and χ are some functions of a and ϕ. The existence of a Noether symmetry induces the existence of the vectors field X such that $\ell_X L = 0$, ℓ_X being the Lie derivative with respect to X. The meaning of this equation is that L is constant along the flow generated by X. We deduce from it a second-degree expression for $\dot{a}$ and $\dot{\phi}$ whose coefficients only depend on a and ϕ and have to vanish.

8.2.1 Vacuum model

Applying the principle described above, we get the following equations when no potential or perfect fluid is present:

$$k(G^{-1}\alpha + (G^{-1})_\phi a \chi) = 0 \tag{8.5}$$

1. Excellent tutorial from professors C. T. Hill and L. M. Lederman can be found on that subject in www.emmynoether.com.

$$G^{-1}\alpha+(G^{-1})_\phi a\chi+2G^{-1}a\frac{\partial\alpha}{\partial a}+(G^{-1})_\phi a^2\frac{\partial\chi}{\partial a}=0 \tag{8.6}$$

$$3\omega\alpha-\omega\phi^{-1}a\chi-6\phi(G^{-1})_\phi\frac{\partial\alpha}{\partial\phi}+2\omega a\frac{\partial\chi}{\partial\phi}+\omega_\phi a\chi=0 \tag{8.7}$$

$$6(G^{-1})_\phi\alpha+3(G^{-1})_{\phi\phi}a\chi+3(G^{-1})_\phi a\frac{\partial\alpha}{\partial a}+6G^{-1}\frac{\partial\alpha}{\partial\phi}-\omega\phi^{-1}a^2\frac{\partial\chi}{\partial a}+3(G^{-1})_\phi a\frac{\partial\chi}{\partial\phi}=0 \tag{8.8}$$

They are the same as these of [106] when we choose $G^{-1}=\phi$. We are going to examine successively the case of curved and flat models.

$k\neq 0$

Following the methods of [106], from the previous equations system, it should be possible to determine a relation between ω and G necessary for the existence of a symmetry and the forms of α and χ determining the vector field generating it. Starting from (8.5), we deduce that $\alpha=-G(G^{-1})_\phi a\chi$. Then, putting this quantity in (8.6) and integrating, we get:

$$\chi = n(\phi)a^{-2} \tag{8.9}$$

$$\alpha = -n(\phi)G(G^{-1})_\phi a^{-1} \tag{8.10}$$

with $n(\phi)$ a function of the scalar field that we have to determine. We introduce these expressions in (8.7) and we get the relation we are looking for between ω and G:

$$\omega=3\omega_0\phi(G^{-1})^2_\phi G(G^{-2}-2\omega_0)^{-1} \tag{8.11}$$

ω_0 being an integration constant. To obtain $n(\phi)$, we replace ω in (8.8) by this last expression. We find that:

$$n(\phi)=n_0(2\omega_0-G^{-2})(G^{-1})^{-1}_\phi \tag{8.12}$$

n_0 being an integration constant. We conclude that in the vacuum case and for a curved FLRW model, a HST whose Brans-Dicke coupling function is linked to the gravitational function by the relation (8.11) has a Noether symmetry generated by a vectors field X defined by (8.12) and (8.9). These relations generalise these found in the case of GST in [106].

$k=0$

The equation (8.5) vanishes and then we have 4 undetermined quantities (the partial derivatives of α and χ) and three equations. To find some solutions, we assume that α and χ can be written with separating variables as successfully done in [106] in the case of GST and we define $\alpha=\alpha_1(a)\alpha_2(\phi)$ and $\chi=\chi_1(a)\chi_2(\phi)$. Introducing these expressions in (8.6), we find that it will be satisfied if $\alpha_2=mG(G^{-1})_\phi\chi_2$, m being a constant. Then we do the same thing with (8.7) and see that we must have $\chi_1=na^{-1}\alpha_1$. Introducing these expressions for α_2 and χ_1 in the equations (8.6-8.8), we deduce from the two first ones the expressions for $\alpha_1^{-1}d\alpha_1/da$ and $\chi_2^{-1}d\chi_2/d\phi$ that we use in the last equations. We get the following relation that have to be satisfied between ω and G:

$$\omega=\phi G(G^{-1}_\phi)^2\frac{6m\omega_0(m+n)G^{3m/n+1}-3}{n^2\omega_0G^{3m/n+1}+2} \tag{8.13}$$

So that α and χ be determined we calculate that:

$$\alpha_1=\alpha_{10}a^{-m(2m+n)^{-1}} \tag{8.14}$$

$$\chi_2=\chi_{20}(G^{-1})^{-1}_\phi G^{\frac{m(3m+2n)}{n(2m+n)}}(n^2\omega_0+2G^{-3m/n-1})^{1/2} \tag{8.15}$$

α_{10} and χ_{20} being integration constants. Consequently a HST whose gravitational function and Brans-Dicke coupling functions are linked by the relation (8.13) have a Noether symmetry generated by the vectors field defined by α_1, α_2, χ_1 and χ_2.

8.2.2 HST with potential

When we consider a potential, only the first equation of (8.5-8.8) changes and is written:

$$6kG^{-1}\alpha - 3a^2U\alpha + 6ka\chi(G^{-1})_\phi - a^3\chi U_\phi = 0 \tag{8.16}$$

Let's consider a model with and without curvature. Using the same method as in section 8.2.1, we express α with (8.16) and put its expression in (8.6) to determine this of χ. It comes:

$$\chi = n(\phi)(2kG^{-1} - a^2U)[3a^2U(G^{-1})_\phi + G^{-1}(6k(G^{-1})_\phi - 2a^2U_\phi)]^{\frac{-3U(G^{-1})_\phi+3U_\phi G^{-1}}{6U(G^{-1})_\phi-4U_\phi G^{-1}}} a^{-2} \tag{8.17}$$

$n(\phi)$ being a function depending on the scalar field. If we Introduce the forms of α and χ in (8.7) we get a differential equation for $n(\phi)$ which is written with 3 different powers of a and logarithm of an expression of a and ϕ. This equation having to vanish, the coefficient of the logarithmic expression has to be equal to zero. This is only possible if the power of the expression for χ is a constant F_0, i.e. when $U = U_0G^{-\frac{3(1+2F_0)}{3+4F_0}}$, U_0 being an integration constant. In the same way the coefficient of the powers of a have to be equal to zero thus defining a system of 3 equations whose the only solution is the General Relativity with $G = G_0$ and $\omega = 0$. However, for this theory χ is undetermined and thus we conclude that for a massive HST in a curved Universe, there is no Noether symmetry. As shown in [103], it does not mean that there is no symmetry at all, and then conserved quantities, but only for the special one we have considered and which belongs to the class of point symmetries[101].

$k = 0$

Contrary to what happens for the vacuum case, the equation (8.5) is not identically zero but is written:

$$3U\alpha + a\chi U_\phi = 0 \tag{8.18}$$

It follows that we have not to assume a variable separation for α and χ. From this last equation, we deduce:

$$\alpha = 1/3a\chi U^{-1}U_\phi \tag{8.19}$$

We introduce this result in (8.6) and derive χ:

$$\chi = n(\phi)a^{\frac{3(U(G^{-1})_\phi - G^{-1}U_\phi)}{-3U(G^{-1})_\phi+2G^{-1}U_\phi}} \tag{8.20}$$

$n(\phi)$ being a function of the scalar field. In the equation (8.8), we replace α and χ by their forms above thus getting a differential equations for $n(\phi)$. Its form is $F_1(\phi)n_\phi + F_2(\phi) + F_3(\phi)ln(a) = 0$. To satisfy it, it is necessary that $F_3 = 0$ thus implying $U = U_0G^{-p}$ with p a constant or $G = G_0$, which corresponds to General Relativity with a massive scalar field. This two cases allow independently that $F_3 = 0$ and are independents each others since in the second one, their is no constraint between G and U. We examine successively these two cases.
When $U = U_0G^{-p}$, we get an expression for n_ϕ from (8.8). Introducing α, χ and n_ϕ in the equation (8.7), we get the following relation between G and ω:

$$\omega = \frac{\phi\left[-3 + 2p(3-p)\omega_0G^{-p+1}\right](G^{-1})^2_\phi}{2G^{-1} - 3\omega_0G^{-p}} \tag{8.21}$$

ω_0 being an integration constant. Using this last expression, we derive the exact form of n:

$$n = n_0G^{\frac{p(-6p^2+p+3)}{2(p-1)(3+2p)^2}}\left(2G^{-1} - 3\omega_0G^{-p}\right)^{\frac{(1+p)(3-2p)^2}{2(p-1)(3+2p)^2}} \tag{8.22}$$

n_0 being an integration constant. Consequently a massive HST whose potential is proportional to a power of the gravitation function which itself depends on the Brans-Dicke coupling function by the relation (8.21) have a Noether symmetry generated by a vectors field X determined by the relations (8.19-8.22).
If we consider the General Relativity, a Noether symmetry only exists in presence of a cosmological constant. However χ is not determined and thus there is no symmetry.
When $G = G_0$, we calculate from the equations (8.7) and (8.8) that a Noether symmetry will exist if:

$$\omega = \frac{2\phi(U_\phi)^2}{G_0(\omega_0 + 3U)U} \tag{8.23}$$

Then, from (8.23), we derive the form of n:

$$n = n_0 U(\omega_0 + 3U)^{1/2}(U_\phi)^{-1} \tag{8.24}$$

Thus a Noether symmetry can exist for General Relativity with a massive scalar field different of a constant in a flat Universe if the relation (8.23) between the potential and the Brans-Dicke coupling function is satisfied and is generated by a vector fields X defined by (8.19), (8.20) et (8.24).

8.2.3 HST with a perfect fluid

The term representing a perfect fluid with an equation of state $p = (\gamma - 1)\rho$ in the Lagrangian is $\rho_0(\gamma - 1)a^{3(1-\gamma)}$. Once again, only the equation (8.5) is modified and written:

$$2kG^{-1}\alpha + 2ak\chi(G^{-1})_\phi - \rho_0(\gamma - 1)^2 a^{2-3\gamma}\alpha \tag{8.25}$$

We deduce from it an expression for α:

$$\alpha = -2ak\chi(G^{-1})_\phi(2kG^{-1} - \rho_0(\gamma - 1)^2 a^{2-3\gamma})^{-1} \tag{8.26}$$

that we introduce in (8.6). Then, we get for χ:

$$\chi = n(\phi)a^{\frac{4-3\gamma}{-2+3\gamma}}\left[(\gamma-1)^2\rho_0 a^2 - 2ka^{3\gamma}G^{-1}\right]\left[(\gamma-1)^2\rho_0 a^2 + 2ka^{3\gamma}G^{-1}\right]^{\frac{1-3\gamma}{-2+3\gamma}} \tag{8.27}$$

$n(\phi)$ being a function depending on the scalar field. We calculate it by using the expressions for α and χ in (8.8). To this end, we consider three values of γ, 1, 0 and 4/3 corresponding respectively to a dust dominated, vacuum energy dominated and radiation dominated Universe. In the first case, the equation (8.25) takes the same form as in the vacuum and thus the results are the same as these of section 8.2.1. In what follows, we will consider only the two last ones.

$k \neq 0$

When $\gamma = 0$ or $\gamma = 4/3$, if we introduce the expressions for α and χ in the equation (8.8), we get a polynomial expression for a whose coefficients have to be zero. It corresponds to a system of 3 equations with 3 unknowns G, ω and n that we have to determined.
When the Universe is vacuum dominated, the only possible solution corresponds to General Relativity or $n = 0$ but once again χ is undetermined and a Noether symmetry does not exist.
When the Universe is radiation dominated, the equations are satisfied if:

$$n = n_0\sqrt{G^{-1}}(G^{-1})_\phi^{-1} \tag{8.28}$$

$$\omega = -3/2\omega_0\phi G(G^{-1})_\phi^2 \tag{8.29}$$

Thus, the HST with a perfect radiative fluid can have a Noether symmetry if this relation between the gravitational coupling function and the Brans-Dicke coupling function is satisfied. It is generated by the vectors field X defined by (8.26-8.28).

$k = 0$

For a flat Universe, the equation (8.25) shows that $\alpha = 0$. Then we can calculate that $\chi = n(\phi)a^{-1}$ whatever $\gamma \neq 1$ with:

$$n(\phi) = n_0\sqrt{\omega_0 - 2G^{-1}}(G^{-1})_\phi \tag{8.30}$$

$$\omega = 3\phi(G^{-1})_\phi^2(\omega_0 - 2G^{-1})^{-1} \tag{8.31}$$

It follows that for a flat model with a perfect fluid, the HST admit a Noether symmetry when the equation (8.31) is satisfied. It is then generated by the vectors field X defined by (8.26), (8.27) and (8.30).

8.3 Discussion

In this work, we have studied the Noether symmetries of the HST for the FLRW models in vacuum, with a massive potential or with a perfect fluid. Our results consist in the determination of conditions allowing the existence of symmetry. They are relations between the functions G, ω or U, conditions on their forms or on the type of geometry. Moreover, for each case, we have determined the vectors field X generating the symmetry.

When we consider a HST without any potential or perfect fluid, for a curved or flat geometry, a Noether symmetry will exist if respectively the relations (8.11) or (8.13) are respected between the gravitational and Brans-Dicke coupling functions. They generalise these found in [101].
When we consider a potential, a symmetry can only exist for a flat Universe. The potential have to be proportional to a power of the gravitational coupling function or the theory to correspond to the General Relativity with a massive scalar field different from a cosmological constant. Respectively, a relation between ω and G defined by (8.21) or ω and U defined by (8.23) have to be satisfied.
When we consider a perfect fluid, for a dust dominated Universe the results are the same as these of the vacuum case. For a curved Universe, if the Universe is vacuum energy dominated, the symmetry does not exist. If it is radiation dominated, a relation between ω and G defined by (8.29) have to be respected. For the existence of a Noether symmetry for a flat Universe and for any type of matter, the relation that is imposed by the symmetry is given by (8.31). These results are summarised in the table 8.1. For each of these cases, we have calculated all the elements allowing the determination of the vectors field X generating the Noether symmetry.

Lets discuss about interesting theories. Whatever the relation existing between G, ω and U for the symmetry existence, the General Relativity defined by $G = G_0$ and $\omega = 0$ with or without a cosmological constant always respects it. However, in this case, the quantity χ is never defined and the General Relativity has no symmetry as conclude in [107]. The only case for which General Relativity admits a Noether symmetry is in presence of a massive scalar field, the potential being different from a cosmological constant, in a flat Universe. Therefore, we recover and generalise the result of [101] which corresponds to the theory defined by $\omega = \phi$, and then to an exponential potential.
If we consider a GST defined by $G^{-1} = \phi$, in the vacuum case, the only GST admitting a Noether theory is defined by $\omega = 3m\omega_0(\phi^2 - 2\omega_0)^{-1}$ for a curved Universe and $\omega = \left[-3 + 6m\omega_0(m+n)\phi^{-3m/n-1}\right](2 + n^2\omega_0\phi^{-3m/n-1})^{-1}$ for a flat Universe as it has been shown in [106] where the dynamics of these two theories are studied. If we consider a potential in a flat Universe, we note that its only form allowing a symmetry is a power law of the scalar field. Then, the Brans-Dicke coupling function have to be $\omega = \left[-3 + 2p(3-p)\omega_0\phi^{p-1}\right](2 - 3\omega_0\phi^{p-1})^{-1}$. It is interesting to note that it is the same form as this of the GST in the empty. If we consider the presence of a radiative fluid, the Brans-Dicke theory is the only GST allowing a symmetry for a Universe with a curvature.
If we consider a theory whose gravitational function is defined by $e^{-\phi}$ and corresponding to the form usually used for the effective string theory action at low energy, we remark that for a flat Universe the only type of potential allowing a symmetry will be an exponential law of the scalar field. The Brans-Dicke coupling function is then $\omega = \phi e^{-\phi}\left[3 + 2p\omega_0(p-3)e^{(p-1)\phi}\right](-2 + 3\omega_0 e^{(p-1)\phi})^{-1}$ and is quite different from the ω usually used for this type of theory although it asymptotically tends toward it.

In a general way, we note that the type of geometry favouring a Noether symmetry is rather a flat one. To our knowledge, such an attempt to draw up a complete list of the HST, in various physical and geometrical contexts, admitting a Noether symmetry, does not exist in the literature. A next step will be to look for conserved quantities and then to try to determine the main dynamical characteristics of classes of models thus defined as it has been done for the special cases of GST in [106]. Here, since our goal was to find the conditions for the existence of Noether symmetries, we have not undertaken this task that will be probably too large for a single article. Another possible extension will be to consider Bianchi models.

	$k \neq 0$	$k = 0$
Empty	$\omega = 3\omega_0\phi(G^{-1})^2_\phi G$ $(G^{-2} - 2\omega_0)^{-1}$	$\omega = \left[-3 + 6m\omega_0\phi^{-3m/n-1}(m+n)\right]$ $(2 + n^2\omega_0\phi^{-3m/n-1})^{-1}$
Potential	No symmetry	If $U = U_0 G^{-p}$, $\omega = \frac{\phi\left[-3+(6-2p)p\omega_0 G^{-p+1}\right](G^{-1})^2_\phi}{2G^{-1}-3\omega_0 G^{-p}}$ If $G = G_0$, $\omega = \frac{2\phi(U_\phi)^2}{G_0(\omega_0+3U)U}$
Dust: $\gamma = 1$	Same as for empty	Same as for empty
Vacuum: $\gamma = 0$	No symmetry	$\omega = 3\phi(G^{-1})^2_\phi(\omega_0 - 2G^{-1})^{-1}$
Radiation: $\gamma = 4/3$	$\omega = -3/2\omega_0\phi G(G^{-1})^2_\phi$	$\omega = 3\phi(G^{-1})^2_\phi(\omega_0 - 2G^{-1})^{-1}$

TAB. 8.1 – *Classification of the Hyperextended Scalar Tensor theories admitting a Noether symmetry.*

Chapitre 9

Conclusion

Nous avons recherché une méthode suffisamment efficace pour nous permettre d'étudier un grand nombre de modèles cosmologiques homogènes afin de contraindre de vastes classes de théories tenseur-scalaires.
De ce point de vue, la moins performante est sans aucun doute celle consistant à rechercher des solutions exactes. Le grand nombre de fonctions inconnues du champ scalaire (fonction de gravitation, fonction de Brans-Dicke, potentiel) et le manque de motivation physique permettant d'en connaître la forme, ne permettent pas de trouver aisément des solutions générales et intéressantes.
Une manière plus intéressante de procéder est de se demander comment contraindre le champ scalaire afin que l'Univers ait certaines propriétés asymptotiques ayant des fondements observationnels ou théoriques solides: expansion, accélération, isotropisation et absence de singularité.
Pour ce faire, nous avons utilisé les formalismes Lagrangien et Hamiltonien. Les résultats sont plus difficiles à obtenir avec le premier qu'avec le second mais les interprétations physiques y sont en revanche plus faciles.
Pour finir, nous avons également considéré la présence d'une symétrie de Noether pour les modèles homogènes et isotropes FLRW. De très fortes contraintes pèsent alors sur les fonctions inconnues des théories tenseur-scalaires mais les motivations physique quant à l'imposition de ces symétries restent obscures. De plus, il a été montré que ces contraintes n'étaient pas toujours compatibles avec les équations de champs.
La plupart de ces méthodes sont soient limitées au modèle de Bianchi de type I le plus simple ou ne permettent d'étudier que des théories tenseur-scalaires avec un nombre limité de fonctions inconnues du champ scalaire.
Dans la partie suivante nous allons utiliser une méthode, comparable à celle décrite par Wainwright et Ellis dans leur livre "Dynamical Systems in Cosmology" mais utilisant le formalisme Hamiltonien ADM au lieu du formalisme orthonormal, qui nous permettra d'étudier n'importe qu'elle théorie tenseur-scalaire pour tous les modèles de Bianchi de la classe A en recherchant les modèles qui s'isotropisent aux époques tardives.

Quatrième partie

Isotropisation des modèles de Bianchi en théories tenseur-scalaires

Nous allons présenter une méthode nous permettant d'étudier le processus d'isotropisation des modèles de Bianchi de la classe A en présence d'un ou plusieurs champs scalaires et de matière. Exiger que ces modèles s'isotropisent nous permettra de contraindre de vastes classes de théories tenseur-scalaires. Contrairement aux autres méthodes que nous avons vues précédemment, il ne sera pas ici nécessaire de se donner de manière adhoc la forme des fonctions du champ scalaire ou de la métrique ou de faire des hypothèses théoriques telles que l'absence d'une singularité initiale ou la présence de symétries de Noether. Nos objectifs sont les suivants:

– Caractériser les champs scalaires capables de conduire un modèle cosmologique anisotrope vers l'isotropie.

  Une théorie tenseur-scalaire peut être définie par plusieurs fonctions du champ scalaire (fonction de Brans-Dicke ω, potentiel U, fonction de gravitation G) dont les formes restent aujourd'hui largement inconnues bien que quelques indices nous soient donnés par la physique des particules comme le mécanisme de Higgs ou la supergravité. Nous verrons qu'imposer un Univers isotrope aux époques tardives est également un moyen de les contraindre.

– Connaître l'état final de l'Univers lorsqu'il est isotrope

  Quel est l'état dynamique de l'Univers lorsque le processus d'isotropisation est achevé? L'isotropisation mène-t-elle à une décélération ou à une accélération de l'expansion? L'Univers est-il plat ou courbé? Est-il dominé par un champ scalaire quintessent? Si nous supposons que le potentiel mime une constante cosmologique variable, peut il résoudre le problème de cette constante?

– Quelle est la robustesse des réponses obtenues aux questions précédentes?

  Afin d'éprouver la robustesse de nos résultats, nous étudierons plusieurs classes de théories tenseur-scalaires. Que se passe-t-il lorsque l'on considère plusieurs champs scalaires, un fluide parfait, un couplage entre ce fluide et un champ scalaire ou un champ scalaire non minimalement couplé à la gravitation?

C'est à cet ensemble de questions que nous allons tenter de répondre en utilisant systématiquement la méthode suivante, mélangeant formalisme Hamiltonien et méthodes d'étude des systèmes dynamiques:

1. On détermine les équations de champs du premier ordre de Hamilton.
2. Afin d'utiliser les méthodes d'études des systèmes dynamiques[25], on réécrira ces équations à l'aide de variables normalisées.
3. On cherchera et étudiera alors leurs points d'équilibre correspondant à des états isotropes stables.
4. On appliquera nos résultats à quelques théories tenseur-scalaires couramment étudiées dans la littérature, ce qui nous permettra de nous assurer de leur validité et de mesurer leur porté.

Dans tout ce qui suit, la métrique que nous utiliserons sera de la forme:

$$ds^2 = -(N^2 - N_i N^i)d\Omega^2 + 2N_i d\Omega\omega^i + R_0^2 g_{ij}\omega^i\omega^j \tag{1}$$

N étant la fonction lapse, N_i les fonctions shifts et ω^i les 1-formes générant les différents espace homogènes de Bianchi. Nous choisirons une métrique diagonale telle que $N_i = 0$ et la relation entre les variables t et Ω sera alors:

$$dt = -Nd\Omega \tag{2}$$

Nous écrirons les fonctions métriques g_{ij} sous la forme:

$$g_{ij} = e^{-2\Omega+2\beta_{ij}}$$

Ω représente alors la partie isotropique de la métrique tandis que les fonctions β_{ij} décrivent sa partie anisotropique. La paramétrisation de Misner[79] permet de réécrire ces fonctions sous la forme:

$$\beta_{ij} = diag(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+) \tag{3}$$

En ce qui concerne l'action, sa forme générale lorsque l'on considère deux champs scalaires minimalement couplés et un fluide parfait sera:

$$S = (16\pi)^{-1}\int[R-(3/2+\omega)\phi^{,\mu}\phi_{,\mu}\phi^{-2}-(3/2+\mu)\psi^{,\mu}\psi_{,\mu}\psi^{-2}- U]\sqrt{-g}d^4x+S_m(g_{ij},\phi,\psi) \tag{4}$$

où $\omega(\phi,\psi)$ et $\mu(\phi,\psi)$ sont deux fonctions de Brans-Dicke décrivant le couplage des champs scalaires avec la métrique et $U(\phi,\psi)$ est le potentiel décrivant le couplage des champs scalaires avec eux même. $S_m(g_{ij},\phi,\psi)$ est l'action représentant la présence d'un fluide parfait non penché éventuellement couplé avec les champs scalaires. Les types de théories tenseur-scalaires que nous allons étudier appartiennent tous à la classe de théorie définie par cette action. Dans ce qui suit, on commence par le cas le plus simple: le modèle de Bianchi de type I dans le vide avec un champ scalaire massif et minimalement couplé.

Chapitre 1

Le modèle de Bianchi de type I(4 articles)

Le modèle de Bianchi de type I est un modèle à sections spatiales plates, géométriquement défini par:

$$\begin{aligned} \omega^1 &= dx \\ \omega^2 &= dy \\ \omega^3 &= dz \end{aligned}$$

Il contient donc les solutions du modèle FLRW à sections spatiales plates. Dans ce chapitre nous allons étudier l'isotropisation de ce modèle lorsque l'on considère un champ scalaire massif minimalement couplé dans le vide, avec un fluide parfait non penché ou avec un second champ scalaire. Pour finir, nous considérerons un champ scalaire non minimalement couplé avec un fluide parfait non penché.

1.1 Dans le vide et avec un seul champ scalaire

Ce cas a été étudié dans l'article [105] reproduit en annexe. Ce fut notre premier article sur le sujet et ses résultats furent améliorés et nuancés dans les articles suivants. Ces modifications sont ici prises en compte.

1.1.1 Equations de champs

Comme nous l'avons montré dans la section 2.3.4 de la partie II, l'Hamiltonien ADM du modèle de Bianchi de type I vide de matière mais avec un champ scalaire minimalement couplé et massif s'écrit:

$$H^2 = p_+^2 + p_-^2 + 12\frac{p_\phi^2\phi^2}{3+2\omega} + 24\pi^2 R_0^6 e^{-6\Omega} U \tag{1.1}$$

Il vient alors pour les équations de Hamilton:

$$\dot{\beta}_\pm = \frac{\partial H}{\partial p_\pm} = \frac{p_\pm}{H} \tag{1.2}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{12\phi^2 p_\phi}{(3+2\omega)H} \tag{1.3}$$

$$\dot{p}_\pm = -\frac{\partial H}{\partial \beta_\pm} = 0 \tag{1.4}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = -12\frac{\phi p_\phi^2}{(3+2\omega)H} + 12\frac{\omega_\phi \phi^2 p_\phi^2}{(3+2\omega)^2 H} - 12\pi^2 R_0^6 \frac{e^{-6\Omega} U_\phi}{H} \tag{1.5}$$

$$\dot{H} = \frac{dH}{d\Omega} = \frac{\partial H}{\partial \Omega} = -72\pi^2 R_0^6 \frac{e^{-6\Omega} U}{H} \tag{1.6}$$

un dot signifiant une dérivée par rapport à Ω. En choisissant $N_i = 0$ et en utilisant le fait que $\partial\sqrt{g}/\partial\Omega = -1/2\Pi_k^k N$ [78], on en déduit la forme de la fonction lapse:

$$N = \frac{12\pi R_0^3 e^{-3\Omega}}{H} \tag{1.7}$$

Afin de trouver les points d'équilibre de ce système d'équations, il nous faut le réécrire partiellement à l'aide de variables normalisées. La forme de l'Hamiltonien ADM nous suggère de définir:

$$x = H^{-1} \tag{1.8}$$

$$y = \pi R_0^3 \sqrt{e^{-6\Omega} U} H^{-1} \tag{1.9}$$

$$z = p_\phi \phi (3 + 2\omega)^{-1/2} H^{-1} \tag{1.10}$$

Ces variables ont toutes une interprétation physique:

- La variable x^2 est proportionnelle à la variable Σ introduite dans [25] et décrit donc le cisaillement(shear).
- La variable y^2 est proportionelle à $(\rho_\phi - p_\phi)/(d\Omega/dt)^2$, $(d\Omega/dt)^2$ représentant la fonction de Hubble lorsque l'Univers est isotrope.
- La variable z^2 est proportionelle à $(\rho_\phi + p_\phi)/(d\Omega/dt)^2$. Pour le montrer, il suffit de remplacer p_ϕ par sa valeur déduite de l'équation pour $\dot{\phi}$
- On déduit des deux derniers points que le paramètre de densité Ω_ϕ du champ scalaire est une combinaison linéaire de y^2 et z^2 ou encore, lorsque le champ scalaire est quintessent, que ces deux variables lui sont proportionelles.

On peut vérifier que ces variables sont normalisées en réécrivant la contrainte Hamiltonienne sous la forme d'une somme de leurs carrés:

$$p^2 x^2 + 24y^2 + 12z^2 = 1 \tag{1.11}$$

avec $p^2 = p_+^2 + p_-^2$. Afin qu'elles soient définies dans l'ensemble des réels, nous considérerons que $3 + 2\omega$ et U sont des quantités positives. La première condition est nécessaire au respect de la condition d'énergie faible. En effet, on peut définir la densité d'énergie et la pression du champ scalaire comme:

$$\rho_\phi = \frac{1}{2}\frac{3/2 + \omega}{\phi^2}\phi'^2 + \frac{1}{2}U$$

$$p_\phi = \frac{1}{2}\frac{3/2 + \omega}{\phi^2}\phi'^2 - \frac{1}{2}U$$

le prime étant une dérivée par rapport au temps propre t. Par conséquent, imposer $3 + 2\omega > 0$ et $U > 0$ revient à écrire que:

$$\rho_\phi + p_\phi > 0$$

$$\rho_\phi - p_\phi > 0$$

la première inégalité étant nécessaire au respect de la condition d'énergie faible. Les équations de Hamilton peuvent être réécrites en fonction de ces variables sous la forme d'un système d'équations différentielles du premier ordre:

$$\dot{x} = 72y^2 x \tag{1.12}$$

$$\dot{y} = y(6\ell z + 72y^2 - 3) \tag{1.13}$$

$$\dot{z} = 24y^2(3z - \frac{1}{2}\ell) \tag{1.14}$$

$$\dot{\phi} = 12z\frac{\phi}{\sqrt{3 + 2\omega}} \tag{1.15}$$

avec $\ell = \phi U_\phi U^{-1}(3 + 2\omega)^{-1/2}$. Nous avons donc réduit les sept équations de Hamilton sous la forme d'un système de quatre équations à quatre inconnues x, y, z et ϕ dont la dernière n'est pas nécessairement normalisée. Cette réduction provient du fait que les équations de Hamilton montrent que $p_\pm$ sont des constantes et que $\beta_+ \propto \beta_-$, éliminant ainsi trois équations sur les sept. Notre prochain objectif est alors de trouver les points d'équilibre correspondant à un état isotrope stable pour l'Univers et dont les propriétés sont définies dans la section suivante.

1.1.2 Définition d'un état isotrope stable

L'isotropie est définie dans l'article de Collins et Hawking[108] lorsque le temps propre tend vers l'infini de la manière suivante:

- $\Omega \to -\infty$
  Cette condition nous dit que l'Univers est en expansion éternelle. Vu qu'aucune période de contraction n'a été observée depuis le découplage rayonnement-matière et que notre Univers est fortement isotrope, cette hypothèse paraît justifiée.
- Soit $T_{\alpha\beta}$ le tenseur d'énergie-impulsion: $T^{00} > 0$ et $\frac{T^{0i}}{T^{00}} \to 0$
  $\frac{T^{0i}}{T^{00}}$ représente une vitesse moyenne de la matière par rapport aux surfaces d'homogénéité. Si cette quantité ne tendait pas vers zéro, l'Univers ne paraîtrait pas homogène et isotrope.
- Soit $\sigma_{ij} = (de^{\beta}/dt)_{k(i}(e^{-\beta})_{j)k}$ et $\sigma^2 = \sigma_{ij}\sigma_{ij}$: $\frac{\sigma}{d\Omega/dt} \to 0$.
  Cette condition dit que l'anisotropie mesurée localement à travers la constante de Hubble tend vers zéro. En effet, lorsque nous mesurons la constante de Hubble, nous évaluons la quantité $\frac{dg_{ii}}{dt}/g_{ii} = d\beta_{ii}/dt - d\Omega/dt$. Pour que celle ci soit la même dans toutes les directions, il faut donc que $d\beta_{ii}/dt << d\Omega/dt$.
- β tend vers une constante β_0
  Cette condition se justifie par le fait que l'anisotropie mesurée dans le CMB est en quelque sorte une mesure du changement de la matrice β entre le temps où la radiation a été émise et le temps où elle a été observée. Si β ne tendait pas vers une constante, on s'attendrait à de grandes quantités d'anisotropies dans certaines directions.

Dans le cadre du modèle de Bianchi de type I, lorsque $\beta_\pm$ tend vers une constante, $d\beta_\pm/dt = -N^{-1}\dot{\beta}_\pm \propto e^{3\Omega}$ tend vers zéro car l'isotropie se produit en $\Omega \to -\infty$. Or, d'une manière générale, lorsque la dérivée d'une fonction tend vers zéro en $t \to +\infty$, ceci n'implique pas nécessairement que la fonction tende vers une constante. C'est par exemple le cas du logarithme $\ln t$. Ceci indique donc que l'isotropie apparait relativement vite. La troisième propriété quant à elle signifie que le cisaillement, c'est-à-dire $x \propto \dot{\beta}_\pm = \frac{d\beta_\pm}{dt}\frac{dt}{d\Omega}$ tend vers zéro. Par conséquent, ***les points d'équilibre isotropes stables que nous recherchons seront tels que:***

$$\Omega \to -\infty$$

$$x \to 0$$

En examinant le système d'équations (1.12-1.14), on constate qu'il existe trois manières d'atteindre cette équilibre correspondant à ***trois classes d'isotropisation*** que l'on définit de la manière suivante:

- Classe 1: Les variables (x,y,z) atteignent un état d'équilibre isotrope avec $y \neq 0$. C'est la classe qui semble correspondre aux théories tenseur-scalaires les plus étudiées dans la littérature.
- Classe 2: Les variables (x,y,z) atteignent un état d'équilibre isotrope avec $y = 0$. Dans ce cas, il n'est généralement pas possible de déterminer le comportement asymptotique de x à l'approche de l'isotropie car il dépend de la manière inconnue dont y tend vers zéro. Or c'est lui qui permet de connaître le comportement asymptotique commun des fonctions métriques lors de l'isotropisation.
- Classe 3: x tend vers l'équilibre mais pas nécessairement les autres variables y et z. Comme celles-ci doivent être bornées en $\Omega \to -\infty$, cela signifie qu'elles doivent osciller telles que leurs dérivées premières par rapport à Ω oscillent autour de zéro. Nous verrons des exemples de cette classe d'isotropisation lorsque nous considérerons la présence de plusieurs champs scalaires.

Dans ce travail notre attention se portera sur l'étude de l'isotropisation de la classe 1. En effet, nous montrerons que les champs scalaires de cette classe sont asymptotiquement quintessents ainsi qu'un lien intéressant avec la présence de matière noire dans les galaxies spirales. Nous verrons quelques exemples numériques d'isotropisation de classe 2 et 3 afin de démontrer leur réalité et de nous permettre de mieux cerner leurs caractéristiques.

1.1.3 Etude des états isotropes

Les points d'équilibre du système (1.12-1.14) tels que $x = 0$ et $y \neq 0$ sont donnés par:

$$(x,y,z) = (0, \pm(3-\ell^2)^{1/2}/(6\sqrt{2}), \ell/6) \tag{1.16}$$

Ils sont définis si $\ell^2 < 3$. y et z devant atteindre l'équilibre, il faut donc que ℓ tende vers une constante nulle ou non et telle que $\dot{\ell} \to 0$. Linéarisant l'équation (1.12) au voisinage de l'équilibre, on trouve qu'asymptotiquement la variable x se comporte en $\Omega \to -\infty$ comme:

$$x \to x_0 e^{3\Omega - \int \ell^2 d\Omega} \tag{1.17}$$

Cadre de validité de nos résultats

Avant d'aller plus loin, ces premiers calculs nous offrent l'opportunité de parler de la stabilité de nos résultats. Ceux ci peuvent être séparés en plusieurs catégories:

1. La localisation des points d'équilibre isotrope.
2. Les conditions nécessaires à leur existence.
3. Les solutions exactes associées aux points d'équilibre.

La première catégorie est indépendante de toute approximation. Les deux autres ne le sont pas et dépendent de la vitesse à laquelle l'état d'équilibre est atteint ou, plus précisément, à laquelle d'une part la fonction ℓ et d'autre part les variables (y,z) tendent vers leurs valeurs à l'équilibre.

En ce qui concerne ℓ, nous verrons dans la troisième partie de cette section qu'il est possible de déterminer asymptotiquement le comportement de $\phi(\Omega)$ et donc de $\ell(\Omega)$. Par conséquent, il est possible de ne faire aucune approximation sur ℓ comme le montre le calcul (1.17) ci-dessus: quelle que soit la vitesse à laquelle ℓ tend vers sa valeur à l'équilibre, la présence du terme $\int \ell^2 d\Omega$ permet de prendre en compte la variation de ℓ^2. Cependant, afin d'obtenir des résultats sous formes fermées et comparables entre eux, nous ferons en général l'hypothèse suivante que nous appellerons "hypothèse de variabilité de ℓ":

– Lorsqu'à l'équilibre ℓ^2 tend vers une constante ℓ_0^2, nulle ou non, avec une variation $\delta\ell^2$ telle que $\ell^2 \to \ell_0^2 + \delta\ell^2$, $\int \ell^2 d\Omega \to \ell_0^2\Omega + const.$

Afin de montrer que l'on peut mathématiquement s'affranchir de cette hypothèse, les résultats de cette section seront tous exprimés en tenant compte de l'intégrale de ℓ^2 puis de l'hypothèse de variabilité de ℓ. De plus, nous appliquerons nos résultats aux cas de deux théories tenseur-scalaires, respectivement en accord et en désaccord avec cette hypothèse. Dans les sections suivantes, elle sera systématiquement employée et vérifiée lorsque nous ferons des applications.

En ce qui concerne y et z, nous considérerons que leurs variations δy et δz à l'approche de l'équilibre sont suffisamment petites pour être négligeable. Par exemple, lorsque nous calculons (1.17), nous prenons en compte la manière dont ℓ approche sa valeur asymptotique puisque nous ne faisons pas l'hypothèse de variabilité de ℓ. En revanche nous ne faisons rien de semblable pour y que nous avons simplement remplacé par sa valeur à l'équilibre dans les équations de champs sans tenir compte de δy. Ce problème ne peut pas être résolu aussi "facilement" que celui en rapport avec ℓ. Il est possible que l'étude des perturbations des solutions exactes puisse apporter des éléments de réponses mais ce n'est pas garanti car elle pourrait fortement dépendre de la spécification des formes de ω et U en fonction du champ scalaire.

Pour résumer, tous nos résultats impliquant le calcul d'une approche asymptotique d'une quantité au voisinage de l'équilibre seront valables lorsque l'Univers atteint suffisamment vite l'état isotrope. La restriction sur ℓ peut être levée en ne faisant pas l'hypothèse de variabilité mais cela semble plus difficile pour les variables y et z. Aussi, nous ferons systématiquement l'hypothèse que ces variables approchent suffisamment vite leurs valeurs à l'équilibre. Ces restrictions seront également valables pour les variables k et w que nous définirons plus tard, respectivement associées à la présence d'un fluide parfait et à celle d'un second champ scalaire ou de courbure.

Comportements asymptotiques

Appliquant l'hypothèse de variabilité de ℓ à (1.17), lorsque ℓ tend vers une constante non nulle, $x \to e^{(3-\ell^2)\Omega}$ et vers $e^{3\Omega}$ sinon. Cette variable disparaît donc bien lorsque $\Omega \to -\infty$ et que la condition de réalité des points d'équilibre, $\ell^2 < 3$, est respectée.
Dans le même temps, l'équation (1.12) montre que x est une fonction monotone de Ω: lorsque x est initialement positive (négative), elle est asymptotiquement croissante(décroissante). x est donc également de signe constant. En se servant de l'expression (1.7) de la fonction lapse N et du fait que $dt = -Nd\Omega$, on en déduit que $\Omega(t)$ est une fonction décroissante (croissante) du temps propre t lorsque x, ou de manière équivalente l'Hamiltonien, est initialement positive (négative). Par conséquent, un Hamiltonien initialement positif est une condition initiale nécessaire pour que l'isotropie en $\Omega \to -\infty$ se produise aux époques tardives. Enfin dernière remarque en se qui concerne les fonctions monotones. Nous pouvons calculer que $dg_{ij}/d\Omega = -2e^{-2\Omega+\beta_{ij}}(1 - \dot{\beta}_{ij})$. Compte tenu de ce que nous avons dit sur la monotonie de x et de l'expression des dérivées de β_{ij} par rapport à Ω, il vient que les β_{ij} sont des fonctions monotones du temps propre t et que par conséquent chaque fonction métrique ne peut avoir au plus qu'un extremum durant son évolution. Nous avions démontré ce point d'une toute autre manière dans [42] à l'aide du formalisme Ha-

miltonien ADM.
Pour déterminer $\phi(\Omega)$, on se sert de l'équation (1.15) pour $\dot{\phi}$ qui s'écrit asymptotiquement:

$$\dot{\phi} = 2\frac{\phi^2 U_\phi}{U(3+2\omega)}$$

Dans cette dernière expression l'hypothèse de variabilité de ℓ n'est pas faite mais par contre on néglige la variation δz de la variable z à l'approche de l'isotropie. C'est la forme asymptotique de la solution de cette équation qui nous donnera le comportement asymptotique de ϕ en fonction de Ω. On en déduira donc $\ell(\Omega)$ et $U(\Omega)$ qui sont 2 fonctions données du champ scalaire. En particulier, connaître $\ell(\Omega)$ permettra de vérifier les conditions nécessaires à l'isotropie, notre hypothèse sur $\int \ell d\Omega$ et donc de calculer la forme asymptotique des fonctions métriques $\Omega(t)$ et du potentiel U. En effet, utilisant d'une part le comportement asymptotique de x et d'autre part la relation $dt = -Nd\Omega$ et la définition de y, on trouve qu'à l'approche de l'isotropie $\Omega(t)$ et U se comportent respectivement comme:

$$dt = -12\pi R_0^3 x_0 e^{-\int \ell^2 d\Omega} d\Omega$$

et

$$U = e^{2\int \ell^2 d\Omega}$$

soit en tenant compte de l'hypothèse de variabilité de ℓ

$$dt = -12\pi R_0^3 x_0 e^{-\ell^2 \Omega} d\Omega$$

et

$$U = e^{2\ell^2 \Omega}$$

Cette hypothèse nous permet de calculer que lorsque ℓ^2 tend vers une constante non nulle, les fonctions métriques tendent vers $t^{\ell^{-2}}$ et le potentiel vers t^{-2}. En revanche, lorsque ℓ^2 tend vers zéro, l'Univers tend vers un modèle de De Sitter et le potentiel vers une constante cosmologique. Si l'hypothèse n'est pas vérifiée, il est impossible de déterminer sans l'aide de quadratures ces comportements asymptotiques.

1.1.4 Discussion et applications

Nos résultats concernent une théorie tenseur-scalaire massive et minimalement couplée sans autre forme de matière. C'est la plus simple des théories que nous allons considérer et la méthode utilisée ci-dessus va nous servir de guide pour les théories suivantes. Nous avons restreint notre étude aux fonctions U et $3+2\omega$ positives et à une isotropisation de classe 1. Lorsque l'on suppose que la fonction ℓ et les variables y et z tendent suffisamment vite vers leurs valeurs à l'équilibre, nous avons les résultats suivants:

Soit une théorie tenseur-scalaire minimalement couplée et massive et la quantité ℓ définie par $\ell = \frac{\phi U_\phi}{U(3+2\omega)^{1/2}}$. Le comportement asymptotique du champ scalaire à l'approche de l'isotropie est donné par la forme de la solution en $\Omega \to -\infty$ de l'équation différentielle $\dot{\phi} = 2\frac{\phi^2 U_\phi}{U(3+2\omega)}$. Une condition nécessaire à l'isotropisation de classe 1 est que $\ell^2 < 3$. Si ℓ tend vers une constante non nulle, les fonctions métriques tendent vers $t^{\ell^{-2}}$ et le potentiel disparaît comme t^{-2}. Si ℓ tend vers zéro, l'Univers tend vers un modèle de De Sitter et le potentiel vers une constante.

En ce qui concerne l'interprétation du nombre 3, on peut montrer intuitivement qu'il est lié à la dimension de l'Univers. En effet, pour l'expliquer, défínsons deux nombres a et b:

- Le premier, a, vaut 6 et provient de l'expression du volume de l'Univers en fonction du facteur d'échelle, $V = R^3 = R^{a/2}$. Il vaut donc deux fois la dimension de l'espace.
- Le second, b, vaut 12 est peut être décomposé en $2*6$. Le 6 provient cette fois de l'écriture du scalaire de courbure à l'aide de la décomposition 3+1 de l'espace-temps. Le 2 est celui du terme cinétique pour le champ scalaire $\dot{\phi}^2$ et apparaît lorsque l'on calcule le moment conjugué de ϕ en variant le Lagrangien par rapport à $\dot{\phi}$.

On trouve alors que le 3 intervenant dans la contrainte $\ell^2 < 3$ nécessaire à l'isotropisation, est défini comme le rapport $a^2/b = 3$ et semble donc clairement lié à la dimension de l'espace que l'on considère.

L'hypothèse de variabilité de ℓ peut être levée et les résultats s'expriment alors à l'aide de l'intégrale de ℓ^2 comme montré ci-dessus. Ils sont en accord avec le "Cosmic No Hair theorem" de Wald[49] qui dit que les modèles homogènes initialement en expansion avec une constante cosmologique positive (sauf le modèle de Bianchi de type IX) et un tenseur d'énergie-impulsion satisfaisant les conditions d'énergies fortes et dominantes tendent vers un modèle isotrope de De Sitter pour lequel l'expansion est exponentielle. Ici, lorsque l'on considère une constante cosmologique ou lorsque $\ell \to 0$ telle que l'hypothèse de variabilité de ℓ est vérifiée, l'Univers lorsqu'il s'isotropise tend bien vers un modèle de De Sitter et le potentiel vers une constante. En revanche, lorsque ℓ tend vers zéro et que l'hypothèse de variabilité de ℓ n'est pas vérifiée, le potentiel ne tend plus vers une constante et l'Univers n'approche plus un modèle de De Sitter.

En guise d'application, nous allons examiner les cas des théories tenseur-scalaires définies par la forme de la fonction de couplage de Brans-Dicke

$$\frac{(3+2\omega)^{1/2}}{\phi} = \sqrt{2}$$

et les formes de potentiels

$$U = e^{m\phi}$$

et

$$U = \phi^m$$

On rappelle que nos résultats représentent des conditions nécessaires et que par conséquent, lorsque dans les applications qui vont suivre nous parlons d'isotropisation, c'est toujours sous réserve que ces conditions soient également suffisantes.

Le potentiel en exponentiel de ϕ possède une longue histoire. L'isotropisation des modèles de Bianchi avec ce potentiel a déjà été étudiée dans [86] et va ainsi nous permettre de tester nos résultats. Il a été montré que tous les modèles de Bianchi(excepté le modèle de Bianchi de type IX lorsqu'il se contracte) s'isotropisaient aux époques tardives lorsque $m^2 < 2$. Si $m = 0$, l'Univers tend vers un modèle de De Sitter car le potentiel est une constante et sinon il est en expansion tel que $e^{-\Omega} \to t^{2m^{-2}}$. Si $m^2 > 2$, les modèles de Bianchi de type I, V, VII et IX peuvent s'isotropiser aux époques tardives. En utilisant nos résultats, nous voyons qu'asymptotiquement:

$$\phi \to m\Omega$$

La condition nécessaire à l'isotropisation de classe 1 s'écrit $m^2 < 6$ et les comportements asymptotiques des fonctions métriques sont bien en accord avec ce qui a été prédit dans [86]. La différence entre les résultats de ce dernier papier et le nôtre porte sur la nature de l'intervalle de m autorisant l'isotropisation puisque nous trouvons une limite supérieure pour celui ci. La figure 1.1 illustre la convergence des variables x, y et z vers leurs valeurs à l'équilibre pour $m = -1$. Lorsque $m^2 > 6$, l'isotropisation de classe 1 n'est plus

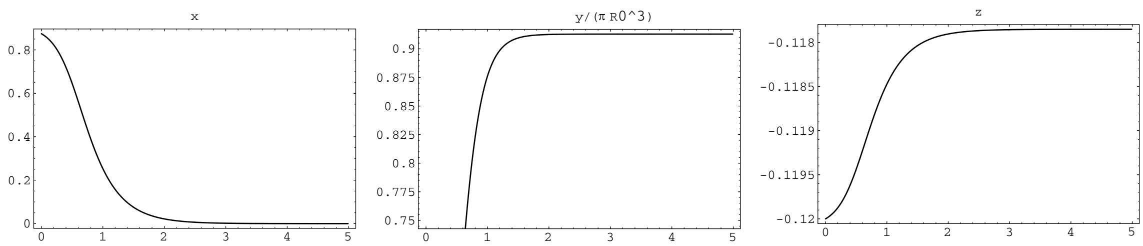

FIG. 1.1 – *Evolution des variables x, y et z lorsque $\frac{(3+2\omega)^{1/2}}{\phi} = \sqrt{2}$, $U = e^{m\phi}$, $R_0^3 = (\sqrt{24}\pi)^{-1}$ et $m = -1$ avec les valeurs initiales $(x,y,z,\phi) = (0.87, 0.25, -0.12, 0.14)$. $\ell = 1/\sqrt{2}$, x tend vers 0, $y(\pi R_0^3)^{-1}$ vers $\sqrt{3-\ell^2}/(6\pi R_0^3\sqrt{2}) = 0.91$ et z vers $\ell/6 = 0.12$ en accord avec l'expression des points d'´equilibre.*

possible car la valeur de y à l'équilibre serait complexe. Une simulation numérique de ce cas est montrée sur la figure 1.2 lorsque $m = -3.2$: l'Univers tend toujours vers un état d'équilibre mais cette fois anisotrope car x tend vers une constante non nulle et donc les fonctions $\beta_\pm$ décrivant l'isotropie, vers l'infini.
Examinons maintenant le cas d'un potentiel en puissance du champ scalaire. On a alors:

$$\ell \to \frac{m}{\sqrt{2}\phi}$$

et à l'approche d'une isotropie de classe 1, si celle-ci se produit, le champ scalaire se comporte comme

$$\phi^2 \to 2m\Omega$$

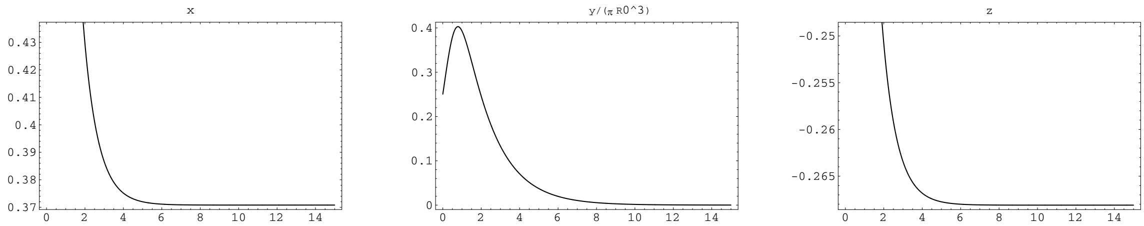

FIG. 1.2 – *Evolution des variables x, y et z lorsque $\frac{(3+2\omega)^{1/2}}{\phi}=\sqrt{2}$, $U=e^{m\phi}$, $R_0^3=(\sqrt{24}\pi)^{-1}$ et $m=-3.2$ avec les valeurs initiales $(x,y,z,\phi)=(0.87,0.25,-0.12,0.14)$. L'Univers ne s'isotropise pas: le système tend vers un point d'équilibre anisotrope tel que les fonctions $\beta_\pm$ décrivant l'anisotropie divergent.*

On doit donc avoir $m<0$ afin que le champ scalaire soit réel et on déduit que ℓ^2 tend vers zéro comme $m(4\Omega)^{-1}$. Par conséquent, $\int \ell^2 d\Omega$ ne tend pas vers une constante mais diverge comme $\frac{m}{4}\ln(-\Omega)$ et nous devons tenir compte de cette intégrale dans nos résultats: l'hypothèse de variabilité de ℓ n'est pas ici vérifiée. Levant cette hypothèse, on trouve alors que le potentiel tend vers zéro comme $(-\Omega)^{m/2}$ et les fonctions métriques vers $exp\left[\left(\frac{4-m}{48\pi R_0^3 x_0}t\right)^{\frac{4}{4-m}}\right]$. Pour que cette quantité diverge positivement, il faut donc que $m<4$ ce qui est toujours vérifié puisque $m<0$. Ce cas est illustré sur la figure 1.3 où l'on voit très nettement que la convergence des variables y et z vers leurs valeurs à l'équilibre est beaucoup plus lente que dans l'application précédente. Ceci signifie que l'Univers approche "lentement" son état isotrope et l'on pourrait alors penser que, en plus de lever l'hypothèse de variabilité de ℓ, les variations δy et δz des variables y et z dont nous parlions dans la sous-section précédente devraient également être prises en compte. Cependant, il semble que ces dernières corrections ne soient pas nécessaires. Ceci peut par exemple être vérifié en comparant l'évolution asymptotique de z correspondant théoriquement à $z\to\ell/6\to-(12\sqrt{-\Omega})^{-1}$ avec l'intégration numérique de la figure 1.3 pour les grandes valeurs de Ω.

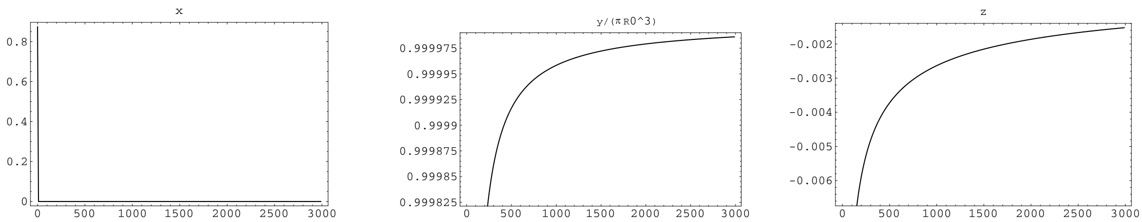

FIG. 1.3 – *Evolution des variables x, y et z lorsque $\frac{(3+2\omega)^{1/2}}{\phi}=\sqrt{2}$, $U=\phi^m$, $R_0^3=(\sqrt{24}\pi)^{-1}$ et $m=-1$ avec les valeurs initiales $(x,y,z,\phi)=(0.87,0.25,-0.12,0.14)$. x tend vers 0, $y(\pi R_0^3)^{-1}$ vers 1 et z vers 0 en accord avec l'expression des points d'équilibre. Remarquons que les variables y et z tendent beaucoup moins vite vers leurs valeurs à l'équilibre que sur le graphe 1.2. Ceci est du à la "lenteur"de la convergence de ℓ vers zéro qui se répercutent alors sur la variation de ces variables.*

Lorsque que $m>0$, une isotropisation de classe 1 ne semble plus possible car alors le champ scalaire serait complexe. Les intégrations numériques semblent indiquer que le champ scalaire devient négatif pour un temps Ω fini. Par conséquent, si l'isotropisation doit se produire en $-\infty$, il semble nécessaire que m soit un entier afin que le potentiel ne soit pas complexe. Les intégrations numériques ne permettent pas d'en dire plus car elles échouent lorsque $\phi\to 0$, signalant peut être la présence d'une singularité.

1.2 Avec fluide parfait et un seul champ scalaire

La démarche est la même que dans le vide mais un terme supplémentaire vient s'ajouter dans les équations de champs[109] dû à la présence d'un fluide parfait d'équation d'état $p=(\gamma-1)\rho$ avec $\gamma\in[1,2]$. Cet intervalle contient les cas importants de la poussière ($\gamma=1$) et de la radiation ($\gamma=4/3$), le cas de la constante cosmologique($\gamma=0$) pouvant être traité avec ce qui a été présenté dans la section précédente. La conservation de l'énergie montre que $\rho\propto V^{-\gamma}$, $V=e^{-3\Omega}$ étant le 3-volume de l'Univers. Nous considérerons que la pression du fluide parfait est isotrope. C'est une hypothèse simplificatrice dont une conséquence pour le modèle de Bianchi de type I pourrait être de rendre la décroissance de l'anisotropie trop rapide (en V^{-1}) pour être détectée et donc observationnellement significative. En effet, la présence d'une pression anisotrope à pour effet de ralentir cette décroissance. L'anisotropie pourrait alors être détectée via le rapport $\delta T/T$ du CMB qui dépend de la quantité $\sigma^2/(d\Omega/dt)$ lors de la surface de dernière diffusion[110].

1.2.1 Equations de champs

Cette fois l'hamiltonien ADM s'écrit:

$$H^2 = p_+^2 + p_-^2 + 12\frac{p_\phi^2\phi^2}{3+2\omega} + 24\pi^2 R_0^6 e^{-6\Omega}U + \delta e^{3(\gamma-2)\Omega} \tag{1.18}$$

où δ est une constante positive. Par rapport au cas de la section précédente, on voit donc apparaître le terme $\delta e^{3(\gamma-2)\Omega}$ dû à la présence du fluide parfait. Les équations de Hamilton deviennent:

$$\dot{\beta}_\pm = \frac{\partial H}{\partial p_\pm} = \frac{p_\pm}{H}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{12\phi^2 p_\phi}{(3+2\omega)H}$$

$$\dot{p}_\pm = -\frac{\partial H}{\partial \beta_\pm} = 0$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = -12\frac{\phi p_\phi^2}{(3+2\omega)H} + 12\frac{\omega_\phi \phi^2 p_\phi^2}{(3+2\omega)^2 H} - 12\pi^2 R_0^6 \frac{e^{-6\Omega}U_\phi}{H}$$

$$\dot{H} = \frac{dH}{d\Omega} = \frac{\partial H}{\partial \Omega} = -72\pi^2 R_0^6 \frac{e^{-6\Omega}U}{H} + 3/2\delta(\gamma-2)\frac{e^{3(\gamma-2)\Omega}}{H}$$

Afin de réécrire ces équations, nous allons nous servir des même variables normalisées x, y et z que dans le vide auxquelles nous ajouterons une quatrième variable:

$$k^2 = \delta e^{3(\gamma-2)\Omega} H^{-2}$$

liée à la présence du fluide parfait. Cette variable est en fait proportionnelle au paramètre de densité du fluide parfait, l'un des paramètres principaux de la cosmologie souvent noté Ω_m. Ceci peut être montré en vérifiant que $k^2 \propto V^{-\gamma}/(\frac{d\Omega}{dt})^2$ où $\frac{d\Omega}{dt}$ est en fait la constante de Hubble lorsque l'Univers s'isotropise. k n'est pas indépendante des trois autres variables et peut se réécrire comme:

$$k^2 = \delta x^\gamma y^{2-\gamma} U^{\gamma/2-1} \tag{1.19}$$

$$k^2 = \delta x^2 e^{3(\gamma-2)\Omega} \tag{1.20}$$

$$k^2 = \delta y^2 U^{-1} V^{-\gamma} \tag{1.21}$$

L'Hamiltonien ADM devient alors:

$$p^2 x^2 + 24y^2 + 12z^2 + k^2 = 1 \tag{1.22}$$

Quant aux équations de Hamilton, elles se réduisent à nouveau à quatre équations:

$$\dot{x} = 72y^2 x - 3/2(\gamma-2)k^2 x \tag{1.23}$$

$$\dot{y} = y(6\ell z + 72y^2 - 3) - 3/2(\gamma-2)k^2 y \tag{1.24}$$

$$\dot{z} = 24y^2(3z - \frac{\ell}{2}) - 3/2(\gamma-2)k^2 z \tag{1.25}$$

$$\dot{\phi} = 12z\frac{\phi}{(3+2\omega)^{1/2}} \tag{1.26}$$

avec toujours $\ell = \phi U_\phi U^{-1}(3+2\omega)^{-1/2}$.

1.2.2 Etude des états isotropes

On distingue deux types d'états d'équilibre selon que k, c'est-à-dire le paramètre de densité du fluide parfait, tend vers zéro ou une constante.

$k \to 0$

Comme y ne tend pas vers zéro car l'on considère une isotropisation de classe 1, on déduit de la forme (1.21) de k que $U >> V^{-\gamma}$. Les points d'équilibre sont les mêmes qu'en l'absence de fluide parfait et donc on retrouve la condition de réalité $\ell^2 < 3$. Le comportement asymptotique de x à l'approche de l'équilibre est obtenu à partir de l'équation (1.23):

$$x \to x_0 e^{3\Omega - \int \ell^2 d\Omega - \frac{3}{2}(\gamma-2)\int k^2 d\Omega} \tag{1.27}$$

De (1.27) et de la définition de y, nous déduisons pour le potentiel:

$$U \to U_0 e^{2\int \ell^2 d\Omega + 3(\gamma-2)\int k^2 d\Omega}$$

En se servant de la définition (1.20) de k et du comportement asymptotique (1.27) de x au voisinage de l'isotropie, il vient:

$$k^2 = \delta x_0^2 e^{-2\int \ell^2 d\Omega - 3(\gamma-2)\int k^2 d\Omega + 3\gamma\Omega}$$

En dérivant cette expression, on obtient l'équation différentielle

$$2k\dot{k} = \left[-2\ell^2 - 3(\gamma-2)k^2 + 3\gamma\right] k^2$$

dont la solution exacte est:

$$k^2 = \frac{e^{3\gamma\Omega - 2\int \ell^2 d\Omega}}{k_0 + 3(\gamma-2)\int e^{3\gamma\Omega - 2\int \ell^2 d\Omega} d\Omega}$$

k_0 étant une constante d'intégration. Tous ces résultats ont été obtenus sans appliquer l'hypothèse de variabilité de ℓ. Si maintenant on la prend en compte, on obtient:

$$k^2 = \delta x_0^2 e^{(-2\ell^2 + 3\gamma)\Omega}$$

et donc que $k \to 0$ en $\Omega \to -\infty$ si $\ell^2 < \frac{3\gamma}{2} < 3$.
Par conséquent la présence d'un fluide parfait telle que $k \to 0$ et le respect de l'hypothèse de variabilité de ℓ réduisent, par rapport au cas du vide, l'intervalle dans lequel doit nécessairement se trouver ℓ^2 pour que l'isotropisation se produise. Cependant, les comportements asymptotiques des fonctions métriques et du potentiel restent inchangés.
Si cette hypothèse n'est pas valide, à nouveau on doit tenir compte des intégrales de ℓ^2. L'intervalle de ℓ^2 permettant l'isotropisation sera toujours tel que $\ell^2 < 3$ mais il sera modifié (ou non) différemment par le condition $k \to 0$. De plus le comportement asymptotique des fonctions métriques et du potentiel sera différent de ce qu'il est dans le vide malgré cette disparition de k.

k tend vers une constante non nulle

Les points d'équilibre isotropes ne sont plus les mêmes et donc les comportements asymptotiques des fonctions métriques et du potentiel non plus. Pour les premiers on trouve:

$$(x,y,z) = (0, \pm\frac{\sqrt{\gamma(2-\gamma)}}{4\sqrt{2}\ell}, \frac{\gamma}{4\ell})$$

après avoir déduit de la contrainte que

$$k^2 = 1 - 3\gamma(2\ell^2)^{-1}$$

Les points d'équilibre seront réels si γ est une constante positive plus petite que 2, en accord avec l'intervalle de variation de γ que nous avons spécifié, soit $\gamma \in [1,2]$. La variable k sera réelle et les autres variables atteindront l'équilibre pour une valeur non nulle de y, respectant ainsi la définition de la classe 1, si ℓ^2 tend vers une constante plus grande que $\frac{3\gamma}{2}$. Cette condition nécessaire à l'isotropie est indépendante de toute approximation. Linéarisant l'équation différentielle pour x, nous trouvons qu'asymptotiquement:

$$x \to e^{\frac{3}{2}(2-\gamma)\Omega}$$

et que les fonctions métriques tendent vers

$$e^{-\Omega} \to t^{\frac{2}{3\gamma}}$$

De la définition de y on déduit alors que le potentiel tend vers zéro comme t^{-2} ce qui est confirmé par la forme (1.21) de k qui montre qu'asymptotiquement:

$$U \propto V^{-\gamma}$$

en accord avec les expressions asymptotiques du potentiel et des fonctions métriques en fonction du temps propre t. Cette dernière expression permet de déterminer la forme asymptotique du champ scalaire d'après la forme du potentiel. Notons que tous ces comportements asymptotiques sont indépendants de l'hypothèse de variabilité de ℓ.

1.2.3 Discussion et applications

Résumons les résultats obtenus en présence d'un fluide parfait. Pour cela, nous allons les énoncer en fonction du paramètre de densité du fluide parfait Ω_m qui est proportionnel à k. Lorsque celui ci tend vers zéro, nous ferons l'hypothèse de variabilité de ℓ alors que celle-ci est inutile lorsque qu'il tend vers une constante non nulle. il vient:

Isotropisation avec $\Omega_m \to 0$:
Les résultats sont les mêmes qu'en l'absence du fluide parfait mais l'intervalle de ℓ^2 permettant l'isotropisation se trouve réduit à $\ell^2 < \frac{3\gamma}{2}$. De plus, lors de l'isotropisation le potentiel du champ scalaire est asymptotiquement supérieur à la densité d'énergie du fluide parfait.

Lorsque l'hypothèse de variabilité de ℓ n'est pas réalisée, les choses ne sont plus aussi simple et les résultats dépendent totalement de la manière dont ℓ^2 approche l'équilibre. On peut cependant toujours les calculer en se servant des expressions données dans les sections précédentes en fonction de $\int \ell^2 d\Omega$.

Lorsque k tend vers une constante non nulle, l'état d'équilibre est différent et nous trouvons que:

Isotropisation avec $\Omega_m \not\to 0$:
Soit une théorie tenseur-scalaire minimalement couplée et massive et la quantité ℓ définie par $\ell = \frac{\phi U_\phi}{U(3+2\omega)^{1/2}}$. Le comportement asymptotique du champ scalaire à l'approche de l'isotropie peut être déduit du fait que $U \propto V^{-\gamma}$: le potentiel du champ scalaire est proportionnel à la densité d'énergie du fluide parfait. Une condition nécessaire à l'isotropisation de classe 1 sera que $\ell^2 > \frac{3\gamma}{2}$, ℓ^2 soit fini et $0 < \gamma < 2$. Alors le potentiel disparaît comme t^{-2} et les fonctions métriques tendent vers $t^{\frac{2}{3\gamma}}$.

Ce dernier résultat montre que la théorie tenseur-scalaire a alors comme attracteur aux époques tardives la Relativité Générale avec un fluide parfait en accord avec [62]. La présence du champ scalaire n'a plus aucun effet sur l'évolution asymptotique des fonctions métriques. Ainsi pour un fluide de poussière tel que $\gamma = 1$, l'Univers tend vers celui d'Einstein-De Sitter avec $e^{-\Omega} \to t^{2/3}$ et pour un fluide radiatif tel que $\gamma = 4/3$, vers un Univers de Tolman avec $e^{-\Omega} \to t^{1/2}$. Remarquons également que les intervalles de ℓ^2 permettant l'isotropisation lorsque $k \to 0$ et $k \not\to 0$ sont complémentaires.
Les applications que nous allons faire concernent les mêmes théories que celles de la section précèdente.
Commençons par considérer un potentiel en exponentiel du champ scalaire. De la même manière qu'en l'absence de fluide parfait, nous obtenons que lorsque $k \to 0$ (respectivement $k \not\to 0$), $m^2 < 3\gamma$ et les fonctions métriques tendent vers $t^{2m^{-2}}$ (respectivement $m^2 > 3\gamma$ et les fonctions métriques tendent vers $t^{\frac{2}{3\gamma}}$). Si $m = 0$, l'Univers tend vers un modèle de De Sitter tel que $k \to 0$. Ces résultats sont en accords avec ceux trouvés dans [111] pour les modèles FLRW. Cependant dans ce dernier papier, une solution stable de type trackers avait aussi été trouvée lorsque $m^2 > 6$. Ici nous ne la retrouvons pas car elle ne permet pas l'isotropie.
En ce qui concerne le potentiel en puissance du champ scalaire lorsque $k \to 0$, nous savons que l'hypothèse de variabilité de ℓ n'est pas vérifiée puisque ℓ tend vers zéro mais que l'intégrale de son carré diverge comme $\frac{m}{4}\ln(-\Omega)$ en $\Omega \to -\infty$ et avec $m < 0$. Tenant compte de cet élément, le calcul de k^2 et de son intégrale nous donne alors:

$$k^2 \to k_0^2(-\Omega)^{-m/2}e^{3\gamma\Omega}$$

et

$$\int k^2 d\Omega \propto \Gamma(1 - m/2, -3\gamma\Omega) \to 0$$

lorsque $\Omega \rightarrow -\infty$, Γ étant la fonction d'Euler. Par conséquent, ces deux quantités tendent vers zéro sans condition supplémentaire et on retrouve les mêmes résultats qu'en l'absence de fluide parfait.
Lorsque k^2 tend vers une constante non nulle, le champ scalaire se comporte comme $\phi \rightarrow e^{-\frac{3\gamma}{m}\Omega}$ et donc ℓ disparaît ou est divergent, interdisant une isotropisation de classe 1.

1.3 Avec un second champ scalaire

Nous considérons désormais deux champs scalaires avec un fluide parfait.
Bien que la plupart des papiers ne prennent en compte qu'un seul champ scalaire, il y a beaucoup de raisons de penser qu'il pourrait y en avoir d'autres. Ainsi la physique des particules prédit l'existence de corrections qui se traduit par l'ajout de termes supplémentaires au scalaire de courbure dans le Lagrangien. Une telle théorie peut être changée via une transformation conforme[112, 113, 26] en une théorie tenseur-scalaire avec plusieurs champs scalaires. Dans les théories supersymétriques, l'ajout de plusieurs champs scalaires permet l'égalité entre les degrés de liberté bosoniques et fermioniques. D'autres raisons sont liées aux théories inflationnaires telle que l'inflation hybride qui nécessite deux champs scalaires[114, 115]: un premier, ψ, décroît vers son minimum local correspondant à un faux vide. Alors l'énergie du vide domine et l'inflation primordiale commence. Pendant ce temps, un second champ scalaire ϕ varie et lorsqu'il atteint une valeur seuil, une variation rapide de ψ se produit. Les deux champs s'ajustent vers des valeurs correspondant à un vrai vide et la fin de l'inflation. Enfin une dernière raison tient à l'existence de champs scalaires complexes. Une théorie tenseur-scalaire avec un champ scalaire complexe ζ peut être transformée en une autre avec deux champs scalaires réels ψ et ϕ à l'aide de la transformation $\zeta = \frac{1}{\sqrt{2m}}\psi e^{im\phi}$.

1.3.1 Equations de champs

L'Hamiltonien ADM d'une théorie tenseur-scalaire avec deux champs scalaires et un fluide parfait s'écrit comme:

$$H^2 = p_+^2 + p_-^2 + 12\frac{p_\phi^2\phi^2}{3+2\omega} + 12\frac{p_\psi^2\psi^2}{3+2\mu} + 24\pi^2 R_0^6 e^{-6\Omega}U + \delta e^{3(\gamma-2)\Omega}$$

et généralise de manière naturelle celui à un champ scalaire. On en déduit les équations de Hamilton:

$$\dot{\beta}_\pm = \frac{\partial H}{\partial p_\pm} = \frac{p_\pm}{H} \tag{1.28}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{12\phi^2 p_\phi}{(3+2\omega)H} \tag{1.29}$$

$$\dot{\psi} = \frac{\partial H}{\partial p_\psi} = \frac{12\psi^2 p_\psi}{(3+2\mu)H} \tag{1.30}$$

$$\dot{p}_\pm = -\frac{\partial H}{\partial \beta_\pm} = 0 \tag{1.31}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = -12\frac{\phi p_\phi^2}{(3+2\omega)H} + 12\frac{\omega_\phi \phi^2 p_\phi^2}{(3+2\omega)^2 H} + 12\frac{\mu_\phi \psi^2 p_\psi^2}{(3+2\mu)^2 H} - 12\pi^2 R_0^6 \frac{e^{-6\Omega}U_\phi}{H} \tag{1.32}$$

$$\dot{p}_\psi = -\frac{\partial H}{\partial \psi} = -12\frac{\psi p_\psi^2}{(3+2\mu)H} + 12\frac{\omega_\psi \phi^2 p_\phi^2}{(3+2\omega)^2 H} + 12\frac{\mu_\psi \psi^2 p_\psi^2}{(3+2\mu)^2 H} - 12\pi^2 R_0^6 \frac{e^{-6\Omega}U_\psi}{H} \tag{1.33}$$

$$\dot{H} = \frac{dH}{d\Omega} = \frac{\partial H}{\partial \Omega} = -72\pi^2 R_0^6 \frac{e^{-6\Omega}U}{H} + 3/2\delta(\gamma-2)\frac{e^{3(\gamma-2)\Omega}}{H} \tag{1.34}$$

On choisit toujours les fonctions shits telles que $N_i = 0$ et la fonction lapse garde la même forme que dans les sections précédentes, soit $N = \frac{12\pi R_0^3 e^{-3\Omega}}{H}$. On va se servir alors des variables suivantes pour réécrire ces équations:

$$x = H^{-1} \tag{1.35}$$

$$y = \pi R_0^3 \sqrt{e^{-6\Omega}U} H^{-1} \tag{1.36}$$

$$z = p_\phi \phi (3+2\omega)^{-1/2} H^{-1} \tag{1.37}$$

$$w = p_\psi \psi (3+2\mu)^{-1/2} H^{-1} \tag{1.38}$$

Comme dans la section précédente, en présence d'un fluide parfait nous définissons la variable k telle que $k^2 = \delta e^{3(\gamma-2)\Omega} H^{-2} = \delta y^2 V^{-\gamma} U^{-1}$. La contrainte Hamiltonienne s'écrit alors:

$$p^2 x^2 + 24 y^2 + 12 z^2 + 12 w^2 + k^2 = 1 \tag{1.39}$$

et les équations de champs:

$$\dot{x} = 72 y^2 x - 3/2(\gamma-2) k^2 x \tag{1.40}$$

$$\dot{y} = y(6\ell_{\phi_1} z + 6\ell_{\psi_1} w + 72 y^2 - 3) - 3/2(\gamma-2) k^2 y \tag{1.41}$$

$$\dot{z} = 24 y^2 (3z - 1/2\ell_{\phi_1}) + 12 w (w \ell_{\phi_2} - z \ell_{\psi_2}) - 3/2(\gamma-2) k^2 z \tag{1.42}$$

$$\dot{w} = 24 y^2 (3w - 1/2\ell_{\psi_1}) + 12 z (z \ell_{\psi_2} - w \ell_{\phi_2}) - 3/2(\gamma-2) k^2 w \tag{1.43}$$

avec

$$\ell_{\phi_1} = \phi U_\phi U^{-1} (3+2\omega)^{-1/2}$$
$$\ell_{\psi_1} = \psi U_\psi U^{-1} (3+2\mu)^{-1/2}$$
$$\ell_{\phi_2} = \phi \mu_\phi (3+2\mu)^{-1} (3+2\omega)^{-1/2}$$
$$\ell_{\psi_2} = \psi \omega_\psi (3+2\omega)^{-1} (3+2\mu)^{-1/2}$$

De plus, les équations de Hamilton pour les champs scalaires sont:

$$\dot{\phi} = 12 z \frac{\phi}{\sqrt{3+2\omega}} \tag{1.44}$$

$$\dot{\psi} = 12 w \frac{\psi}{\sqrt{3+2\mu}} \tag{1.45}$$

Dans ce qui suit nous adopterons l'hypothèse de variabilité aux quatre fonctions ℓ. Ceci permet d'alléger considérablement les calculs, sachant que cette hypothèse peut être levée comme prescrit dans la section 1. Nous allons étudier deux familles de théories tenseur-scalaires:

- La première est telle que ω et μ dépendent respectivement de ϕ et ψ seulement, c'est-à-dire $\ell_{\phi_2} = \ell_{\psi_2} = 0$ alors que U pourra dépendre des deux champs scalaires. Donc le couplage entre ϕ et ψ n'apparaît qu'à travers le potentiel. Ce type de théories est souvent l'aboutissement de la compactification d'espace-temps de dimensions supérieures à quatre.
- La seconde est telle que U et μ ne dépendent que de ψ alors que ω dépend des deux champs scalaires. Nous aurons alors $\ell_{\phi_1} = \ell_{\phi_2} = 0$. Ces caractéristiques résultent de la transformation d'un Lagrangien avec un champ scalaire complexe en un autre Lagrangien avec deux champs scalaires réels

Chacune de ces théories sera étudiée avec et sans fluide parfait.

1.3.2 Sans fluide parfait

Dans cette partie, on considère que $k = 0$ strictement, c'est-à-dire l'absence de fluide parfait et nous examinons successivement les deux cas décrits ci-dessus, à savoir $\ell_{\phi_2} = \ell_{\psi_2} = 0$ et $\ell_{\phi_1} = \ell_{\phi_2} = 0$.

$\underline{\ell_{\phi_2} = \ell_{\psi_2} = 0}$

Nous commençons par calculer les points d'équilibre compatibles avec une isotropisation de classe 1. Nous trouvons:

$$(x,y,z,w) = (0, \pm (3 - \ell_{\phi_1}^2 - \ell_{\psi_1}^2)^{1/2} (\sqrt{3} R)^{-1}, \ell_{\phi_1}/6, \ell_{\psi_1}/6)$$

Ils sont réels si $\ell_{\phi_1}^2 + \ell_{\psi_1}^2 < 3$ et permettent aux variables y et z d'atteindre l'équilibre et d'être bornées si ℓ_{ϕ_1} et ℓ_{ψ_1} tendent vers des constantes.

En ce qui concerne les fonctions monotones, on peut à nouveau montrer que Ω est une fonction monotone du temps propre t telle que $\Omega \to -\infty$ correspond aux époques tardives si l'Hamiltonien est initialement positif[1].

Pour les comportements asymptotiques des fonctions, on montre en utilisant l'hypothèse de variabilité appliquée à ℓ_{ϕ_1} et ℓ_{ψ_1} que:

$$x \to x_0 e^{(3 - \ell_{\phi_1}^2 - \ell_{\psi_1}^2)\Omega}$$

1. Pour des d´etails techniques voir [116] reproduit en annexe

Cette quantité tend bien vers zéro en $\Omega \to -\infty$ lorsque la condition de réalité des points d'équilibre est respectée. En se servant de l'expression de la fonction lapse et de la relation $dt = -Nd\Omega$, on trouve qu'asymptotiquement les fonctions métriques tendront vers:

$$e^{-\Omega} \to t^{(\ell_{\phi_1}^2+\ell_{\psi_1}^2)^{-1}}$$

lorsque $\ell_{\phi_1}^2 + \ell_{\psi_1}^2$ tend vers une constante non nulle ou vers une exponentielle du temps propre sinon: le potentiel tend alors respectivement vers t^{-2} ou une constante.
Les formes asymptotiques des champs scalaires correspondent aux solutions asymptotiques des équations couplées du premier ordre:

$$\dot{\phi} = \frac{2\phi^2 U_\phi}{(3+2\omega)U}$$

$$\dot{\psi} = \frac{2\psi^2 U_\psi}{(3+2\mu)U}$$

L'ensemble de ces résultats généralise ceux trouvés en présence d'un unique champ scalaire.

$\underline{\ell_{\phi_1} = \ell_{\phi_2} = 0}$

Comme nous allons le voir, les choses sont ici complètement différentes. Tout d'abord on trouve deux points d'équilibre pouvant correspondre à un état isotrope stable:

$$\begin{aligned} E_1 &= (0, \pm(1-\ell_{\psi_1}^2/3)^{1/2}R^{-1}, 0, \ell_{\psi_1}/6) \\ E_2 &= (0, \pm\left[2\ell_{\psi_2}(\ell_{\psi_1}+2\ell_{\psi_2})^{-1}\right]^{1/2}R^{-1}, \\ & \pm(\ell_{\psi_1}^2 + 2\ell_{\psi_1}\ell_{\psi_2} - 3)^{1/2}\left[2\sqrt{3}(\ell_{\psi_1}+2\ell_{\psi_2})\right]^{-1}, \\ & (2\ell_{\psi_1}+4\ell_{\psi_2})^{-1}) \end{aligned}$$

Le premier sera réel et fini si $\ell_{\psi_1}^2 \leq 3$ et tend vers une constante. Pour le second, il faut que $\ell_{\psi_2}(\ell_{\psi_1} + 2\ell_{\psi_2})^{-1}$ tende vers une constante positive, $\ell_{\psi_1}(\ell_{\psi_1}+2\ell_{\psi_2}) \geq 3$ et $\ell_{\psi_1} + 2\ell_{\psi_2} \neq 0$. Notons que pour E_2, ℓ_{ψ_1} et ℓ_{ψ_2} ne sont pas nécessairement finis.
On peut dire la même chose que plus haut sur la monotonie de la fonction Ω par rapport au temps propre t.

Pour le premier point d'équilibre, on trouve que le comportement asymptotique de z est:

$$z \to e^{(3-\ell_{\psi_1}^2)\Omega - 2\int \ell_{\psi_1}\ell_{\psi_2}d\Omega}$$

Une intégrale apparaît dans cette équation car l'expression des points d'équilibre n'impose aucune contrainte à ℓ_{ψ_2} qui peut par exemple diverger. Elle montre que nous devons avoir

$$(3-\ell_{\psi_1}^2)\Omega - 2\int \ell_{\psi_1}\ell_{\psi_2}d\Omega \to -\infty$$

afin que z disparaisse. De plus, si l'on considère l'équation (1.43), on remarque la présence du terme $z^2\ell_{\psi_2}$. On en déduit que z doit disparaître suffisamment vite pour permettre à w d'atteindre l'équilibre et ainsi contrer une éventuelle divergence de ℓ_{ψ_2}, soit

$$z^2\ell_{\psi_2} \to 0$$

La variable x se comporte quant à elle comme:

$$x_0 e^{(3-\ell_{\psi_1}^2)\Omega}$$

c'est-à-dire comme en présence d'un unique champ scalaire. Nous verrons ce que cela signifie physiquement dans la section 2.3. Elle disparaît bien en $\Omega \to -\infty$ lorsque la condition de réalité des points d'équilibre est respectée. Comme précédemment et en appliquant l'hypothèse de variabilité à ℓ_{ψ_1}, on obtient que si l'isotropisation se produit, les fonctions métriques tendent vers

$$e^{-\Omega} \to t^{\ell_{\psi_1}^{-2}}$$

lorsque ℓ_{ψ_1} tend vers une constante non nulle ou vers une exponentielle du temps propre sinon. Le potentiel tend alors respectivement vers t^{-2} ou une constante.
Les formes asymptotiques des champs scalaires correspondent aux solutions asymptotiques des équations couplées du premier ordre:

$$\dot{\phi} = 12\phi(3+2\omega)^{-1/2} e^{(3-\ell_{\psi_1}^2)\Omega - 2\int \ell_{\psi_1}\ell_{\psi_2} d\Omega}$$

$$\dot{\psi} = \frac{2\psi^2 U_\psi}{(3+2\mu)U} \tag{1.46}$$

Pour le second point d'équilibre, appliquant l'hypothèse de variabilité à $\ell_{\psi_2}(\ell_{\psi_1}+2\ell_{\psi_2})^{-1}$, on trouve pour le comportement asymptotique de x:

$$x \rightarrow x_0 e^{3\left[2\ell_{\psi_2}(\ell_{\psi_1}+2\ell_{\psi_2})^{-1}\right]\Omega}$$

Puisque $2\ell_{\psi_2}(\ell_{\psi_1}+2\ell_{\psi_2})^{-1}$ tend vers une constante positive, x disparaît bien en $\Omega \rightarrow -\infty$. Alors lorsque $1 - 2\ell_{\psi_2}(\ell_{\psi_1}+2\ell_{\psi_2})^{-1} = \ell_{\psi_1}(\ell_{\psi_1}+2\ell_{\psi_2})^{-1}$ tend vers une constante non nulle, les fonctions métriques tendent vers:

$$e^{-\Omega} \rightarrow t^{(\ell_{\psi_1}+2\ell_{\psi_2})(3\ell_{\psi_1})^{-1}}$$

ou vers une exponentielle du temps propre sinon. A partir des conditions de réalités du point E_2, il est possible de vérifier que cette puissance de t est positive, en accord avec la croissance de $e^{-\Omega}$ lorsque $\Omega \rightarrow -\infty$. A nouveau le potentiel tend vers t^{-2} ou une constante selon que $\ell_{\psi_1}(\ell_{\psi_1}+2\ell_{\psi_2})^{-1}$ tend vers une constante non nulle ou nulle. Quant aux champs scalaires, leurs comportements asymptotiques sont ceux des solutions du système d'équations différentiels du premier ordre:

$$\dot{\phi} = -2\sqrt{3}\frac{\phi}{\psi}\frac{\sqrt{-3U^2(3+2\mu)(3+2\omega)+\psi^2 U_\psi\left[U(3+2\omega)\right]_\psi}}{\left[U(3+2\omega)\right]_\psi}$$

$$\dot{\psi} = \frac{6U(3+2\omega)}{\left[U(3+2\omega)\right]_\psi} \tag{1.47}$$

Cette dernière équation s'intègre pour donner $U(3+2\omega) = e^{6(\Omega-\Omega_0)}$, Ω_0 étant une constante d'intégration.

1.3.3 Avec fluide parfait

Comme en présence d'un seul champ scalaire nous allons scinder cette section en deux parties, selon que le paramètre de densité du fluide parfait tend vers zéro ou une constante non nulle au voisinage de l'isotropie.

$\underline{k \rightarrow 0}$

Comme nous l'avons déjà vu, les points d'équilibre et les comportements asymptotiques des fonctions métriques sont les mêmes qu'en l'absence de fluide parfait et $U << V^{-\gamma}$. En revanche, le fait de supposer que $k \rightarrow 0$ ajoute une contrainte supplémentaire.
Pour le cas où $\ell_{\phi_2} = \ell_{\psi_2} = 0$, la nouvelle contrainte généralise celle trouvée en présence d'un seul champ scalaire avec un fluide parfait: k tendra asymptotiquement vers zéro si $\ell_{\phi_1}^2 + \ell_{\psi_1}^2 < 3/2\gamma$.
En ce qui concerne le cas pour lequel $\ell_{\phi_1} = \ell_{\phi_2} = 0$ et le point E_1, la nouvelle contrainte nécessaire à la disparition de k sera $\ell_{\psi_1}^2 < 3/2\gamma$ et pour le point E_2, $2\ell_{\psi_2}(\ell_{\psi_1+2\ell_{\psi_2}})^{-1} > 1 - \gamma/2$. Les démonstrations peuvent être trouvées dans l'article [116] reproduit en annexe.

$\underline{k \not\rightarrow 0}$

Ce cas implique qu'asymptotiquement $U \propto V^{-\gamma}$. Les points d'équilibre sont alors différents de ceux trouvés lorsque la variable k est nulle ou disparaît asymptotiquement.
Lorsque $\ell_{\phi_2} = \ell_{\psi_2} = 0$, les points d'équilibre correspondant à une isotropisation de classe 1 sont:

$$\begin{aligned} E_{4,5} \quad = \quad & (0, \pm 1/2\sqrt{3}R^{-1}\left[\gamma(2-\gamma)(\ell_{\phi 1}^2+\ell_{\psi 1}^2)^{-1}\right]^{1/2}, 1/4\gamma\ell_{\phi 1}(\ell_{\phi 1}^2+\ell_{\psi 1}^2)^{-1}, \\ & 1/4\gamma\ell_{\psi 1}(\ell_{\phi 1}^2+\ell_{\psi 1}^2)^{-1}) \end{aligned}$$

avec $k^2 \to 1 - \frac{3\gamma}{2(\ell_{\phi 1}^2+\ell_{\psi 1}^2)}$ qui est réel et non nul si $\ell_{\phi 1}^2 + \ell_{\psi 1}^2 > 3/2\gamma$. L'équilibre sera atteint si $\ell_{\phi 1}$ et $\ell_{\psi 1}$ tendent vers des constantes. On montre alors que les fonctions métriques tendent vers $t^{\frac{2}{3\gamma}}$, le potentiel vers t^{-2} et que le champ scalaire se comporte asymptotiquement comme la solution en $\Omega \to -\infty$ de

$$\dot{\phi} = 3\gamma \frac{(3+2\mu)\phi^2 U U_\phi}{(3+2\mu)\phi^2 U_\phi^2 + (3+2\omega)\psi^2 U_\psi^2} \tag{1.48}$$

$$\dot{\psi} = 3\gamma \frac{(3+2\omega)\phi\psi U U_\psi}{(3+2\mu)\phi^2 U_\phi^2 + (3+2\omega)\psi^2 U_\psi^2} \tag{1.49}$$

Lorsque $\ell_{\phi_1} = \ell_{\phi_2} = 0$, les points d'équilibre correspondant à une isotropisation de classe 1 sont:

$$E_{2,3} = (0, \pm 1/2R^{-1}\ell_{\psi 1}^{-1}\sqrt{3\gamma(2-\gamma)}, 0, 1/4\gamma\ell_{\psi 1}^{-1})$$

avec $k^2 \to 1 - 3/2\gamma\ell_{\psi_1}^{-2}$. La réalité de k implique donc que $\ell_{\psi 1}^2 > 3/2\gamma$. De plus, afin d'atteindre un état d'équilibre correspondant à une isotropisation de classe 1 telle que $y \neq 0$, il est nécessaire que $\ell_{\psi 1}$ tende vers une constante. Les fonctions métriques et le potentiel tendent alors respectivement vers $t^{\frac{2}{3\gamma}}$ et t^{-2}. Le comportement asymptotique du champ scalaire ψ peut être déterminé grâce au fait que $U(\psi) \propto V^{-\gamma}$ et celui du champ scalaire ϕ par la relation

$$\dot{\phi} = \phi_0 \frac{12\phi}{\sqrt{3+2\omega}} e^{3\left[(1-\gamma/2)\Omega - \gamma\int \ell_{\psi 2}\ell_{\psi 1}^{-1} d\Omega\right]}$$

Dans tous ces calculs pour lesquels $k \not\to 0$, aucune hypothèse de variabilité n'a été faite.

1.3.4 Discussion

Résumons nos résultats. Nous rappelons qu'ils concernent une isotropisation de classe 1 et que les comportements asymptotiques des fonctions ont été établis en supposant, sauf autrement précisé, des hypothèses de variabilité et que les variables (y,z,w) tendent suffisamment vite vers leurs valeurs à l'équilibre.

Cas A: Sans fluide parfait:

Cas 1A: $\omega(\phi)$, $\mu(\psi)$ et $U(\phi,\psi)$
Une condition nécessaire pour l'isotropisation du modèle de Bianchi de type I lorsque deux champs scalaires minimalement couplés et massifs sont présents sera que les deux quantités $\ell_{\phi_1} = \phi U_\phi U^{-1}(3+2\omega)^{-1/2}$ et $\ell_{\psi_1} = \psi U_\psi U^{-1}(3+2\mu)^{-1/2}$ tendent vers des constantes telles que $\ell_{\phi_1}^2 + \ell_{\psi_1}^2 < 3$. Lorsque l'isotropisation se produit et que l'une des deux constantes est non nulle, les fonctions métriques tendent vers $t^{(\ell_{\phi_1}^2+\ell_{\psi_1}^2)^{-1}}$ et le potentiel vers t^{-2}. Si les deux constantes disparaissent, l'Univers tend vers un modèle de De Sitter et le potentiel vers une constante.

Si l'on pose $\ell_{\psi_1} = 0$, on retrouve les mêmes résultats qu'en présence d'un seul champ scalaire. Ceux ci peuvent être généralisés à la présence de n champs scalaires ϕ_i dont la fonction de Brans-Dicke ω_i ne dépend uniquement que du champ ϕ_i(voir l'annexe 1 de l'article [116] reproduit en annexe). Pour cela, il est suffisant de remplacer $\ell_{\phi_1}^2 + \ell_{\psi_1}^2$ par la somme $\sum_i \ell_i^2$. Dans la littérature, il a été montré que la présence de plusieurs champs scalaires pouvait favoriser l'inflation. C'est ce que l'on appelle l'inflation assistée[117]. L'inverse a aussi été montré: plus il y a de champs scalaires, moins l'inflation a de chances de se produire[117]. Il semble que ce soit ce dernier comportement qui arrive lors de l'isotropisation: plus il y a de champs scalaires, plus ils contribuent au dénominateur de la puissance du temps vers laquelle tendent les fonctions métriques, et moins de chance elle aura d'être supérieure à l'unité et de permettre un comportement accéléré de la métrique.

Cas 2A: $\omega(\phi,\psi)$, $\mu(\psi)$ et $U(\psi)$
Il existe deux points d'équilibre E_1 et E_2 qui peuvent correspondre à un état d'équilibre isotrope pour le modèle de Bianchi de type I lorsque deux champs scalaires minimalement couplés et massifs sont présents. Les conditions nécessaires pour atteindre l'équilibre sont exprimées à l'aide des deux quantités $\ell_{\psi_1} = \psi U_\psi U^{-1}(3+2\mu)^{-1/2}$ et $\ell_{\psi_2} = \psi\omega_\psi(3+2\omega)^{-1}(3+2\mu)^{-1/2}$:

- *Pour le point E_1, il est nécessaire que $\ell_{\psi_1}^2 < 3$ et $(3-\ell_{\psi_1}^2)\Omega - 2\int \ell_{\psi_1}\ell_{\psi_2} d\Omega \to -\infty$.*
  Lorsque l'isotropisation se produit et si ℓ_{ψ_1} tend vers une constante non nulle, les fonctions métriques

tendent vers $t^{\ell_{\psi_1}^{-2}}$ *et le potentiel disparaît comme* t^{-2}.
Lorsque l'isotropisation se produit et si ℓ_{ψ_1} *tend vers zéro, l'Univers tend vers un modèle de De Sitter et le potentiel vers une constante. Si de plus* ℓ_{ψ_2} *diverge, une condition supplémentaire pour l'isotropisation est que* $\ell_{\psi_2} e^{2\left[(3-\ell_{\psi_1}^2)\Omega - 2\int \ell_{\psi_1}\ell_{\psi_2} d\Omega\right]} \rightarrow 0$.

– *Pour le point* E_2, *il est nécessaire que* $0 < 2\ell_{\psi_2}(\ell_{\psi_1} + 2\ell_{\psi_2})^{-1} < 1$, $\ell_{\psi_1} + 2\ell_{\psi_2} \neq 0$ *et* $\ell_{\psi_1}(\ell_{\psi_1} + 2\ell_{\psi_2}) > 3$.
Lorsque l'isotropisation se produit et si $\ell_{\psi_1}(\ell_{\psi_1} + 2\ell_{\psi_2})^{-1}$ *tend vers une constante non nulle, les fonctions métriques tendent vers* $t^{(\ell_{\psi_1}+2\ell_{\psi_2})(3\ell_{\psi_1})^{-1}}$ *et le potentiel disparaît comme* t^{-2}.
Lorsque l'isotropisation se produit et si $\ell_{\psi_1}(\ell_{\psi_1} + 2\ell_{\psi_2})^{-1}$ *tend vers zéro, l'Univers tend vers un modèle de De Sitter et le potentiel vers une constante.*

Pour ce second type de théories lié aux champs scalaires complexes, il existe donc deux points d'équilibre ce qui n'est jamais le cas avec un unique champ scalaire réel. Pour le premier point d'équilibre, le comportement asymptotique des fonctions métriques ne dépend que de ψ alors que pour le second, il dépend des deux champs scalaires.

Cas B: Avec fluide parfait:

A nouveau les résultats dépendent du fait que k tende ou non vers une constante différente de zéro.

Case 1B: $\omega(\phi)$, $\mu(\psi)$ *et* $U(\phi,\psi)$.
Une condition nécessaire pour l'isotropisation du modèle de Bianchi de type I *lorsque deux champs scalaires minimalement couplés et massifs sont présents et tels que* $U \propto V^{-\gamma}$ ($\Omega_m \not\rightarrow 0$) *sera que les quantités* $\ell_{\phi_1} = \phi U_\phi U^{-1}(3+2\omega)^{-1/2}$ *et* $\ell_{\psi_1} = \psi U_\psi U^{-1}(3+2\mu)^{-1/2}$ *tendent vers des constantes telles que* $\ell_{\phi_1}^2 + \ell_{\psi_1}^2 > 3/2\gamma$. *Alors, lorsque l'isotropisation se produit les fonctions métriques tendent vers* $t^{\frac{2}{3\gamma}}$ *et le potentiel disparaît comme* t^{-2}. *Lorsque l'isotropisation se produit telle que* $U >> V^{-\gamma}$ ($\Omega_m \rightarrow 0$), *nous retrouvons les mêmes résultats que dans le cas 1A mais la condition sur* $\ell_{\phi_1}^2 + \ell_{\psi_1}^2$ *est transformée en* $\ell_{\phi_1}^2 + \ell_{\psi_1}^2 < 3/2\gamma$.

Lorsque $k \rightarrow const \neq 0$, le comportement asymptotique des fonctions métriques est le même qu'en présence d'un seul champ scalaire, montrant la stabilité de ce résultat vis à vis de la présence d'un second champ. Si maintenant nous considérons le second type de couplage en relation avec des champs scalaires complexes, nous avons:

Cas 2B: $\omega(\phi,\psi)$, $\mu(\psi)$ *et* $U(\psi)$.
Soient les quantités $\ell_{\psi_1} = \psi U_\psi U^{-1}(3+2\mu)^{-1/2}$ *et* $\ell_{\psi_2} = \psi\omega_\psi(3+2\omega)^{-1}(3+2\mu)^{-1/2}$. *Des conditions nécessaires pour l'isotropisation du modèle de Bianchi de type* I *lorsque deux champs scalaires minimalement couplés et massifs sont présents et tels que* $U \propto V^{-\gamma}$ ($k \rightarrow const \neq 0$) *seront que* ℓ_{ψ_1} *tend vers une constante telle que* $\ell_{\psi_1}^2 > 3/2\gamma$ *et* $(1-\gamma/2)\Omega - \gamma\int \ell_{\psi 2}\ell_{\psi 1}^{-1} d\Omega \rightarrow -\infty$ *lorsque* $\Omega \rightarrow -\infty$. *Lorsque l'isotropisation se produit, les fonctions métriques tendent vers* $t^{\frac{2}{3\gamma}}$ *et le potentiel disparaît comme* t^{-2}. *Lorsque l'isotropisation se produit telle que* $U >> V^{-\gamma}$ ($k \rightarrow 0$), *nous retrouverons les mêmes résultats que pour le cas* $2A$ *mais les conditions nécessaires pour l'isotropisation vers les points d'équilibre* E_1 *et* E_2 *sont respectivement transformées en* $\ell_{\psi_1}^2 < 3/2\gamma$ *et* $1-\gamma/2 < 2\ell_{\psi_2}(\ell_{\psi_1+2\ell_{\psi_2}})^{-1} < 1$.

1.3.5 Applications

Afin d'illustrer nos résultats, nous allons examiner les conditions de l'isotropisation de quelques théories étudiées dans la littérature.

Inflation hybride

Au début de la section 1.3, nous avons expliqué le lien entre les théories tenseur-scalaires avec deux champs scalaires et l'inflation hybride . L'inflation hybride a entre autre été étudiée dans [114] avec une théorie tenseur-scalaire définie par:

$$(3+2\omega)\phi^{-2} = 2 \quad (1.50)$$

$$(3+2\mu)\psi^{-2} = 2 \quad (1.51)$$

$$U = 1/4\lambda(\psi^2 - M^2) + 1/2m^2\phi^2 + 1/2\lambda'\phi^2\psi^2 \quad (1.52)$$

m, M, λ et λ' étant des constantes. Cette théorie correspond aux cas $1A$ et $1B$ définis dans la discussion. Le même type de théorie est également utilisé dans [115] du point de vue des défauts topologiques. Pour un modèle FLRW avec section spatiale plate, l'inflation s'arrête quand l'état de vrai vide, correspondant au minimum global du potentiel en $(\phi,\psi) = (0,M)$, est atteint. Lorsque aucun fluide parfait n'est présent, on calcule que ℓ_{ϕ_1} et ℓ_{ψ_1} sont respectivement proportionnels à $\dot{\phi}$ et $\dot{\psi}$ et s'écrivent:

$$\ell_{\phi_1} = \frac{2\sqrt{2}\phi(m^2 + \lambda'\psi^2)}{\lambda(M^2 - \psi^2)^2 + 2\phi^2(m^2 + \lambda'\psi^2)} \tag{1.53}$$

$$\ell_{\psi_1} = \frac{2\sqrt{2}\psi\left[\lambda'\phi^2 + \lambda(\psi^2 - M^2)\right]}{\lambda(M^2 - \psi^2)^2 + 2\phi^2(m^2 + \lambda'\psi^2)} \tag{1.54}$$

Lorsque $(\phi,\psi) = (0,M)$, nous avons de manière évidente $\phi \to 0$ et $M^2 - \psi^2 \to 0$. Alors, si l'on suppose que la disparition de ϕ est plus petite, plus rapide ou du même ordre que $M^2 - \psi^2$, nous trouvons respectivement que ℓ_{ϕ_1}, ℓ_{ψ_1} ou le couple $(\ell_{\phi_1},\ell_{\psi_1})$ divergent. Donc, il en est de même pour les dérivées des champs scalaires. Par conséquent, le couple $(\phi,\psi) = (0,M)$ représente un état asymptotique de vrais vide qui ne peut se produire lors d'une isotropisation de classe 1 du modèle de Bianchi de type I.
En présence d'un fluide parfait, des simulations numériques indiquent que ϕ oscille vers zéro alors que ψ tend vers une constante M_0 différente de M lorsque $\Omega \to -\infty$. Donc le potentiel tend vers une constante et non vers $V^{-\gamma}$. Par conséquent, l'isotropisation ne se produit pas lorsque $k \neq 0$. Puisqu'elle ne peut pas non plus arriver en l'absence de fluide parfait, nous concluons à l'absence d'isotropisation de classe 1 également lorsque $k \to 0$.
Donc, l'isotropisation de classe 1 semble impossible pour la théorie définie ci-dessus. Des simulations numériques effectuées sur le système (1.40-1.43) confirme ce résultat et ne montre pas non plus d'isotropisation de classe 2 ou 3.

Théories d'ordre supérieure et compactifi cation

Une autre théorie peut être définie par les même formes de fonctions de couplage de Brans-Dicke mais avec un autre potentiel:

$$U = U_0 e^{-\sqrt{2/3}n\phi} e^{-5\sqrt{3}/6n\psi}(e^{\sqrt{3}/2\psi} - 1)^m \tag{1.55}$$

avec $n > 0$ et $m > 0$ [2]. De tels potentiels apparaissent lorsque l'on compactifie l'espace-temps et transforme une théorie d'ordre supérieur pour le scalaire de Ricci en une forme relativiste. Ainsi dans [113], une transformation conforme est appliquée à la théorie définie par $S = \int d^5x\sqrt{G_5}(\frac{M_5^3}{16\pi}R_5 + \alpha M_5^{-3}R_5^4)$ et permet d'obtenir la théorie tenseur-scalaire ci-dessus avec $m = 4/3$, alors que si l'on considère l'action $S = \int d^5x\sqrt{G_5}(\frac{M_5^3}{16\pi}R_5 + bM_5R_5^2 + cM_5^{-3}R_5^4)$, cela correspond cette fois à $m = 2$. Ces actions sont liées à la compactification de la théorie M. En l'absence de fluide parfait, utilisant les comportements asymptotiques des champs scalaires, nous trouvons que près de l'isotropie:

$$\phi \to -\sqrt{2/3}n\Omega \tag{1.56}$$

$$-\sqrt{2/3}n\Omega + \phi_0 \to -\frac{2\sqrt{2}}{5(5n-3m)}\left[2\sqrt{3}m\ln\left[e^{\sqrt{3}\psi/2}(5n-3m) - 5n\right] + (5n-3m)\psi\right] \tag{1.57}$$

Puisque $n > 0$, ψ ne diverge pas vers $-\infty$ autrement le membre de gauche de l'équation (1.57) serait complexe. Les simulations numériques montrent que ψ tend vers $+\infty$ lorsque $\Omega \to -\infty$ et nous déduisons alors de (1.57) que $\psi \to -(5n-3m)(2\sqrt{3})^{-1}\Omega$. Cette limite se produira en $\Omega \to -\infty$ si $5n - 3m > 0$. Nous calculons que les quantités ℓ_{ϕ_1} et ℓ_{ψ_1} tendent respectivement vers les constantes $-n/\sqrt{3}$ et $(3m - 5n)(2\sqrt{6})$. La condition nécessaire à l'isotropisation est ainsi $(11n^2 - 10nm + 3m^2)/8 < 3$. Supposant que $(n,m) \neq (0,0)$, le comportement asymptotique des fonctions métriques à l'approche de l'équilibre isotrope est $t^{24\left[8n^2+(5n-3m)^2\right]^{-1}}$. Ainsi, après des transformations conformes, ces théories issues de la physique des particules peuvent conduire à une isotropisation de classe 1 du modèle de Bianchi de type I comme illustré sur la figure 1.4.
Lorsqu'un fluide parfait est présent, les analyses numériques montrent que ψ est défini en $\Omega \to -\infty$ et que

2. Ces suppositions permettent de simplifi er l'´etude.

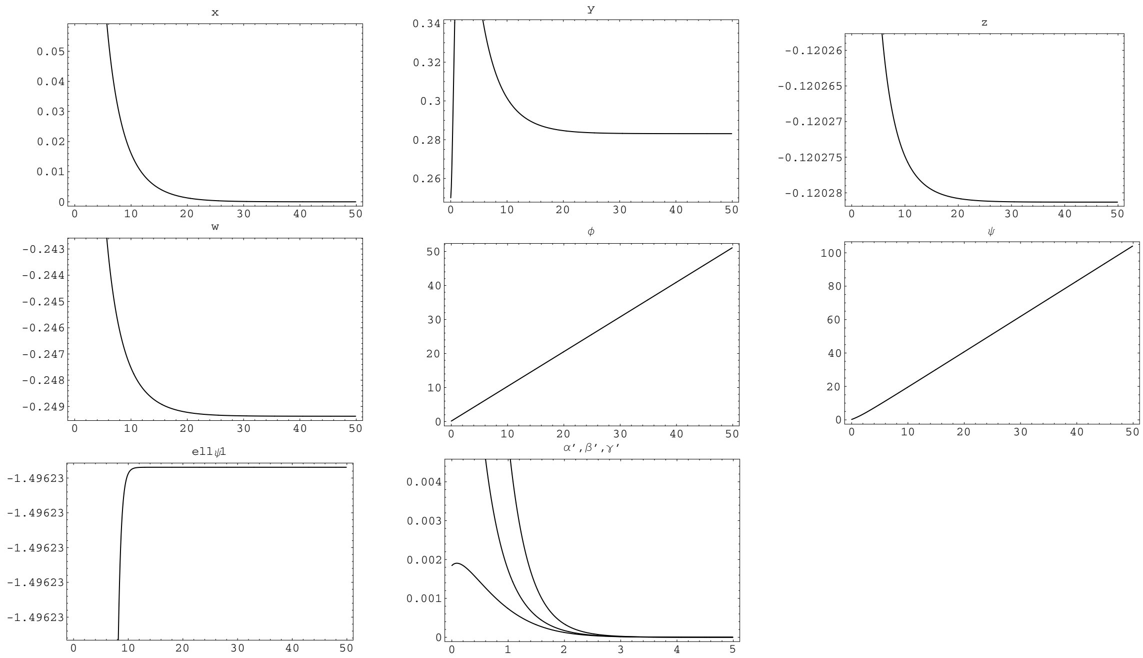

FIG. 1.4 – *Ces figures, avec $-\Omega$ en abscisse, représentent successivement les comportements de $(x,y,z,w,\phi,\psi,\ell_{\psi_1})$ pour les conditions initiales $(x,y,z,w,\phi,\psi) = (-0.49,0.25,-0.12,-0.15,0.14,0.23)$ et les paramètres $(U_0,n,m) = (3.2,1.25,-0.36)$. ϕ et ψ se comportent alors respectivement au voisinage de l'isotropie comme -1.02Ω et -2.12Ω. Notons que ℓ_{ϕ_1} est une constante $-n/\sqrt{3} = -0.721688$. La dernière figure montre la disparition des dérivées de α, β et γ par rapport au temps propre comme cela doit être en cas de convergence vers une puissance du temps propre. Si nous choisissons $m = -2.36$, $(11n^2 - 10nm + 3m^2)/8 = 7.92 > 3$ et l'isotropisation de classe 1 ne se produit pas car x tend vers une constante non nulle*

ce champ scalaire devrait diverger. De la forme de $\dot{\phi}$ et $\dot{\psi}$, on voit que ψ ne peut pas tendre vers $-\infty$ pour un n positif lorsque $\Omega \to -\infty$. Quand $\psi \to +\infty$, il vient que $\ell_{\phi_1}^2 + \ell_{\psi_1}^2 \to (11n^2 - 10nm + 3m^2)/8$ et donc cette théorie peut s'isotropiser vers un état d'équilibre dont la nature (c'est-à-dire le fait que k tende ou non vers une constante nulle) dépend de la valeur de cette constante par rapport à $3/2\gamma$. Ce cas est illustré sur le figure 1.5 où une intégration numérique a été effectuée avec $(11n^2 - 10nm + 3m^2)/8 > 3/2\gamma$. Des intégrations numériques des champs scalaires produisent également des solutions pour lesquelles ψ tend vers zéro et ϕ tend vers une constante non nulle, mais alors $\ell_{\phi_1}^2 + \ell_{\psi_1}^2$ diverge et une isotropisation de classe 1 est impossible.

Champ scalaire complexe avec potentiel quadratique

Les théories correspondant aux cas $2A$ et $2B$ peuvent être liées à la présence d'un champ scalaire complexe dont le Lagrangien prend généralement la forme[118, 119, 120]:

$$L = R + g^{\mu\nu}\zeta^*_{,\mu}\zeta_{,\nu} - V(|\zeta|^2) + L_m \tag{1.58}$$

En redéfinissant le champ scalaire ζ comme $\zeta = \psi(\sqrt{2}m)e^{-im\phi}$, il vient:

$$L = R + 1/2g^{\mu\nu}(\psi^2\phi_{,\mu}\phi_{,\nu} + m^{-2}\psi_{,\mu}\psi_{,\nu}) - U(\psi^2) + L_m \tag{1.59}$$

ce qui correspond à $3/2 + \mu = 1/2m^{-2}\psi^2$ et $3/2 + \omega = 1/2\phi^2\psi^2$. Le potentiel dépendant de ψ^2, sa forme la plus simple et la plus naturelle semble être $U = \zeta\zeta^* = \psi^2$. Cette forme est souvent utilisée par exemple pour la quantification du champ scalaire dans [118] ou pour étudier si l'inflation est générique pour les modèles spatialement fermés[120].

Si l'on suppose qu'il n'y a pas de fluide parfait, alors pour le point d'équilibre E_1, nous obtenons que $\psi \to \pm 2m\sqrt{2(\Omega - \psi_0)}$: ce champ est complexe lorsque $\Omega \to -\infty$ alors que, par définition, il devrait être réel. Pour le point d'équilibre E_2, nous obtenons $\psi \to \psi_0 e^{3/2\Omega}$ alors que maintenant ϕ tend vers une valeur complexe au lieu d'une valeur réelle. Par conséquent, pour la théorie définie par (1.59) avec $U = \psi^2$, une isotropisation de classe 1 est impossible. Cependant, les simulations numériques portant sur les équations (1.40-1.41) révèle que l'Univers devrait subir une isotropisation de classe 3 comme montré sur la figure 1.6

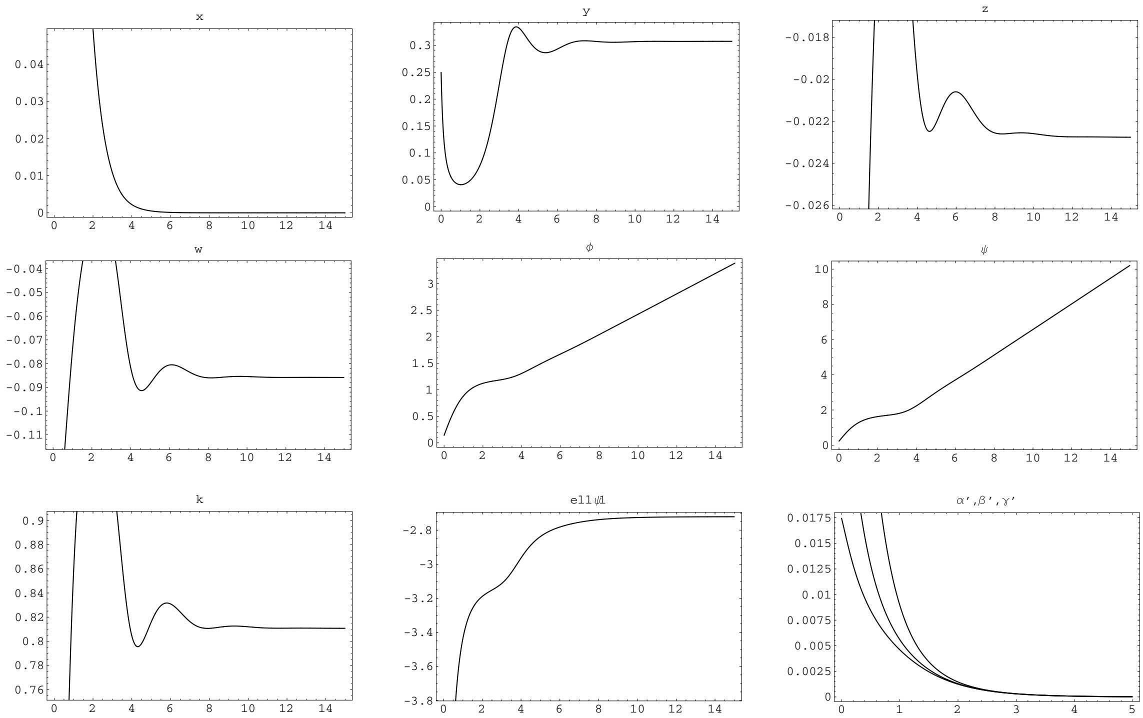

FIG. 1.5 – *Ces figures, avec $-\Omega$ en abscisse, représentent successivement les comportements de $(x,y,z,w,\phi,\psi,k,\ell_{\phi_1})$ pour les conditions initiales $(x,y,z,w,\phi,\psi) = (-0.49,0.25,-0.12,-0.15,0.14,0.23)$ et les paramètres $(U_0,n,m) = (3.2,1.25,-2.36)$ avec un fluide de poussière. Notons que ℓ_{ϕ_1} est une constante valant $-n/\sqrt{3} = -0.721688$. Comme précédemment, la dernière figure montre les dérivées de α, β et γ par rapport au temps propre.*

et avec les caractéristique énoncées à la section 1.1.2 pour cette classe.

Si maintenant on suppose la présence d'un fluide parfait et que l'on considère le cas tel que $k \neq 0$, on calcule que le champ scalaire ψ tend vers $e^{3/2\gamma\Omega}$ et donc ℓ_{ψ_1} diverge comme $e^{-3/2\gamma\Omega}$: l'isotropisation de classe 1 est impossible. Cependant, une fois de plus les intégrations numériques montrent qu'une isotropisation de classe 3 est possible avec k oscillant vers une constante comme montré sur la figure 1.7. Si $k \to 0$, une isotropisation de classe 1 est impossible pour les mêmes raisons qu'en l'absence de matière au contraire d'une isotropisation de classe 3.

Défauts topologiques

Un autre type de potentiel a été utilisé dans [121] pour étudier la formation de défauts topologiques après l'inflation. Sa forme est $U = \lambda/2(\psi^2 - \eta^2)^2$ avec λ et η des constantes.

En l'absence de fluide parfait, nous calculons pour le point E_1 que $\psi^2 \to -\eta^2 ProductLog(-\eta^{-2}\, e^{-16m^2\eta^{-2}(\Omega-\phi_0)})$, ϕ_0 étant une constante[3]. Mais cette quantité est négative lorsque $\Omega \to -\infty$ et donc une fois de plus ψ est asymptotiquement complexe. Pour le point E_2, nous trouvons aussi que ψ est complexe lorsque $\Omega \to -\infty$ sauf si la constante d'intégration est elle même complexe. Ainsi, quelque soit le point d'équilibre E_1 et E_2, un état d'équilibre isotrope de classe 1 ne peut se produire car au moins l'un des champs scalaires est complexe aux époques tardives.

Supposons la présence d'un fluide parfait tel que $k \neq 0$, nous avons alors $\psi^2 \to e^{3/2\gamma(\Omega-\Omega_0)} + \eta^2$. Par conséquent, ℓ_{ψ_1} diverge et une isotropisation de classe 1 ne se produit pas pour la même raison que dans l'application précédente. Comme elle n'arrive pas non plus dans le cas du vide, il en est de même si $k \to 0$. Cependant, une fois de plus, nous avons observé une isotropisation de classe 3 avec et sans matière. En présence de matière, k tend vers une constante avec des oscillations amorties et nous avons observé que x mais aussi z et les champs scalaires peuvent atteindre l'équilibre. Ceci est illustré par la figure 1.8. Les mêmes remarques s'appliquent en l'absence de matière et, globalement, les comportements des fonctions sont les mêmes que ceux montrés sur la figure 1.6.

3. $ProducLog(z)$ donne la solution principale de w dans $z = we^w$.

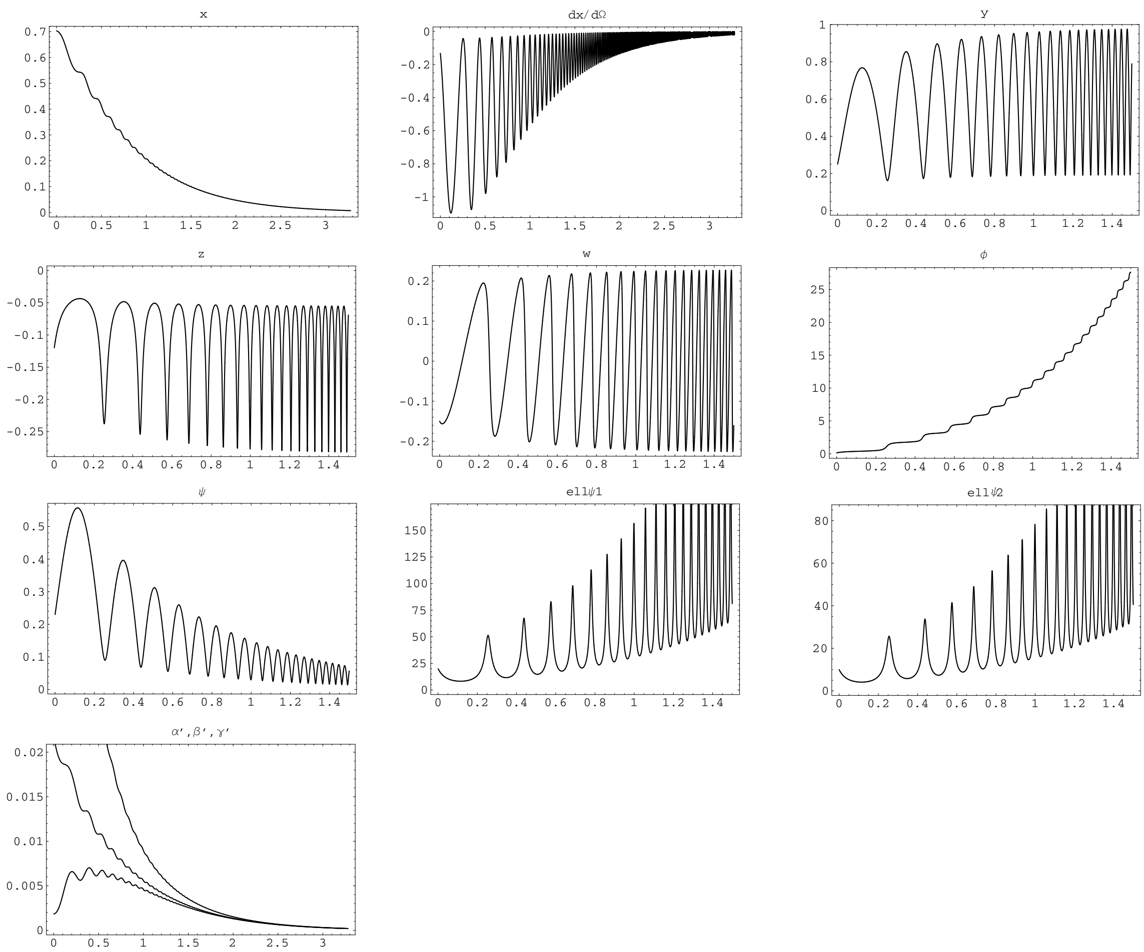

FIG. 1.6 – *Ces figures, avec $-\Omega$ en abscisse, représente successivement les comportements de $(x,\dot{x},y,z,w,\phi,\psi,\ell_{\phi_2},\ell_{\psi_2})$ pour les conditions initiales $(x,y,z,w,\phi,\psi) = (-0.70,0.25,-0.12,-0.15,0.14,0.23)$ et le paramètre $m = -2.3$. x est la seule variable à atteindre l'équilibre alors que y, z et w oscillent de plus en plus lorsque $-\Omega \to +\infty$. Le champ scalaire subit des oscillations amorties alors que les oscillations de ℓ_{ϕ_2} et ℓ_{ψ_2} augmentent. La dernière figure montre la disparition des dérivées de α, β et γ par rapport au temps propre. Notons qu'elles oscillent.*

Condensat de Bose-Einstein

Dans [122], un condensat de Bose-Einstein est étudié[4] avec un potentiel de la forme $\alpha\psi^2 + \beta\psi^4$.

En l'absence de fluide parfait, ψ est complexe pour le point d'équilibre E_1. En fait, $\psi \to \left[\alpha(2\beta^{-1})\right]^{1/2}$ $(ProductLog(\alpha^{-1}e^{1+32m^2\beta\alpha^{-1}(\Omega-\Omega_0)}) - 1)^{1/2}$ avec Ω_0 une constante d'intégration. Ainsi, lorsque $\Omega \to -\infty$, la seconde racine carrée est réelle si $\alpha\beta^{-1} < 0$ mais alors la première est complexe.
Pour le point d'équilibre E_2, ψ^2 tend vers une constante $-\alpha\beta^{-1}$ avec $\alpha < 0$ et $\beta > 0$. Dans le même temps, $\phi \to -2(-3\beta\alpha^{-1})^{1/2}\Omega + \phi_0$, ϕ_0 étant une constante d'intégration. Calculant ℓ_{ψ_1} et ℓ_{ψ_2}, nous obtenons respectivement que ℓ_{ψ_1} diverge et $\ell_{\psi_2} \to \pm m\sqrt{-\beta\alpha^{-1}}$. Ainsi, $2\ell_{\psi_2}(\ell_{\psi_1} + 2\ell_{\psi_2})^{-1} \to 0$ et $y \to 0$. Nous pourrions donc avoir une isotropisation de classe 2 bien que les simulations numériques aient échoué à la montrer.

Si maintenant on considère la présence d'un fluide parfait tel que $k \not\to 0$, nous trouvons que $\psi^2 \to -\alpha(2\beta)^{-1} \pm (2\beta)^{-1}(\alpha^2 + 4\beta e^{-3\gamma(\Omega_0-\Omega)})^{1/2}$, Ω_0 étant une constante d'intégration. Alors, ℓ_{ψ_1} diverge et une isotropisation de classe 1 n'est pas possible. En revanche, les résultats obtenus dans le vide montrent qu'un état isotrope stable peut être atteint lorsque $k \to 0$ et pour le point E_2. Il nous faut alors nous assurer que k tend bien vers zéro. Or sa disparition nécessite que $1 - \gamma/2 < 2\ell_{\psi_2}(\ell_{\psi_{1+2\ell_{\psi_2}}})^{-1}$, ce qui est toujours vrai car le membre de droite de cette inégalité tend bien vers zéro. Nous concluons donc que la théorie devrait subir une isotropisation de classe 1 mais nous ne l'avons pas observé numériquement(nous rappelons que nous avons trouvé des conditions nécessaires et non suffisantes à l'isotropisation).

4. Le Lagrangian est différent de (1.58).

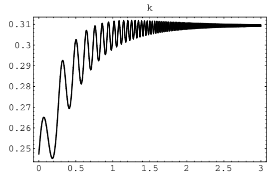

FIG. 1.7 – *Si l'on prend en compte un fluide parfait, k peut atteindre une constante durant l'isotropisation.*

Une fois de plus, les simulations numériques montrent une isotropisation de classe 3 avec et sans fluide parfait et avec les mêmes comportements que ceux montrés sur la figure 1.6.

Nous observons que toutes les théories ayant un champ scalaire complexe semblent atteindre l'isotropie via une isotropisation de classe 3 alors que les autres l'atteignent via la classe 1. Ceci pourrait être dû au fait que nous avons principalement considéré des théories avec champ scalaire complexe telles que $U \propto \psi^2+\psi^4$ et ne doit donc pas être considéré comme une règle.

1.4 Avec champ scalaire non minimalement couplé

Dans cette section, nous allons étudier l'isotropisation d'un modèle de Bianchi de type I pour une théorie tenseur-scalaire non minimalement couplée et définie par

$$L = (G^{-1}R - \omega\phi^{-1}\phi_{,\mu}\phi^{,\mu} - U + T^{\alpha\beta}\delta g_{\alpha\beta})\sqrt{g} \tag{1.60}$$

Ce type de théorie est aussi connu sous le nom de théorie tenseur-scalaire hyperétendue (HST)[35]. Pour cela, nous allons nous servir de la transformation conforme suivante pour la métrique $g_{\alpha\beta}$

$$g_{\alpha\beta} = G\bar{g}_{\alpha\beta} \tag{1.61}$$

$$dt = \sqrt{G}d\bar{t}$$

qui change le Lagrangien ci-dessus en

$$L = \left[\bar{R} - (3/2(G^{-1})^2_\phi G^2 + \omega G\phi^{-1})\phi_{,\mu}\phi^{,\mu} - G^2U + G^3T^{\alpha\beta}\delta\bar{g}_{\alpha\beta}\right]\sqrt{\bar{g}} \tag{1.62}$$

Les quantités barrées sont alors les quantités du référentiel d'Einstein défini par les fonctions métriques $\bar{g}_{\alpha\beta}$ et celles non barrées sont les quantités du référentiel de Brans-Dicke défini par les fonctions métriques $g_{\alpha\beta}$. Dans ce Lagrangien, le champ scalaire est minimalement couplé à la courbure mais non minimalement couplé à la matière. Il implique que la matière ne suit pas les géodésiques de l'espace temps. De plus, la loi habituelle de conservation de l'énergie-impulsion n'est pas respectée et il nous faut la réévaluer afin d'obtenir le terme H_m de l'Hamiltonien ADM représentant la matière. Elle a entre autre été calculée dans [123] et [124]. Nous avons les relations suivantes concernant les tenseurs d'énergie-impulsion des référentiels de Brans-Dicke et d'Einstein:

$$\begin{aligned}\bar{T}^{\alpha\beta} &= G^3T^{\alpha\beta}\\ \bar{T} &= G^2T\end{aligned}$$

et qui nous permettent d'en déduire la loi de conservation suivante:

$$\begin{aligned}\bar{T}^{\alpha\beta}_{;\alpha} &= 3G_{,\alpha}G^2T^{\alpha\beta} \text{ (since } T^{\alpha\beta}_{;\alpha}=0)\\ \bar{T}^{\alpha\beta}_{;\alpha} &= 3G_{,\alpha}G^2g^{\alpha\beta}T^{\alpha}_{\alpha}\\ \bar{T}^{\alpha\beta}_{;\alpha} &= 3G_{,\alpha}G^2G^{-1}\bar{g}^{\alpha\beta}G^{-2}\bar{T}\\ \bar{T}^{\alpha\beta}_{;\alpha} &= 3G_{,\alpha}G^{-1}\bar{g}^{\alpha\beta}\bar{T}\\ \bar{T}^{\alpha\beta}_{;\alpha} &= -3\frac{dG}{dt}G^{-1}\bar{T} \text{ (puisque } G=G(t))\end{aligned}$$

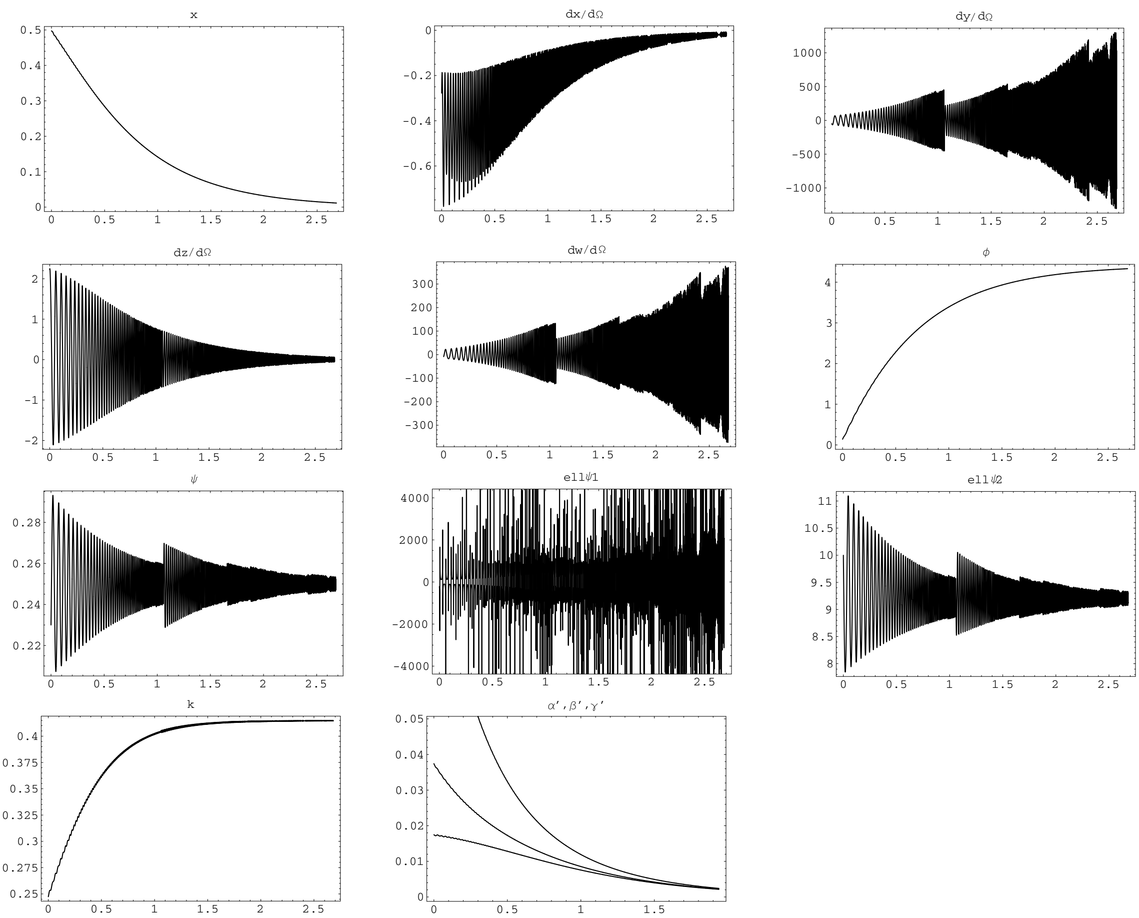

FIG. 1.8 – *Ces fi gures, avec $-\Omega$ en abscisse, repr´esentent successivement les comportements de $(x,\dot{x},\dot{y},\dot{z},\dot{w},\phi,\psi,\ell_{\psi_2},\ell_{\psi_2})$ pour les valeurs initiales $(x,y,z,w,\phi,\psi) = (0.49,0.25,-0.12,-0.15,0.14,0.23)$ et les param`etres $(\lambda,\eta) = (0.25,0.25)$. $x$, $z$ et le champ scalaire atteignent l'´equilibre alors que $\ell_{\psi_1}$ subit des oscillations non amorties. La derni`ere fi gure montre la disparition des d´eriv´ees de α, β et γ par rapport au temps propre.*

Dans [124], cette loi est interprétée comme l'action d'une force sur la matière due à la variabilité des masses au repos. Afin de simplifier les calculs, on pose $p^* = G^2 p$ et $\rho^* = G^2\rho$. Ainsi, nous avons $\bar{T}^{\alpha\beta} = (\rho^* + p^*)u^\alpha u^\beta + \bar{g}^{\alpha\beta}p$. De plus, l'équation d'état est de la forme $p = (\gamma - 1)\rho$ et donc il vient:

$$\begin{aligned}
\bar{T}^{0\beta}_{;\beta} &= -3\frac{dG}{dt}G^{-1}(3p^* - \rho^*) \\
\frac{d\rho^*}{dt} + (\rho^* + p^*)V^{-1}\frac{dV}{dt} &= -3\frac{dG}{dt}G^{-1}(3\gamma - 4)\rho^* \\
\rho^{*-1}\frac{d\rho^*}{dt} + \gamma V^{-1}\frac{dV}{dt} &= -3\frac{dG}{dt}G^{-1}(3\gamma - 4) \\
\rho^* V^\gamma = G^{3(4-3\gamma)}
\end{aligned}$$

De cette dernière expression et de la forme du Lagrangien pour le fluide parfait[125, pages 48-52], nous déduisons pour L_m:

$$\begin{aligned}
L_m &= T^{\alpha\beta}\delta g_{\alpha\beta}\sqrt{g} \\
&= -8\pi R_0^3 N e^{-3\Omega}\rho \\
&= -8\pi R_0^3 \bar{N} e^{-3\bar{\Omega}}\rho^* \\
&= -8\pi R_0^3 \bar{N} e^{-3\bar{\Omega}} G^{3(4-3\gamma)} V^{-\gamma}
\end{aligned}$$

et par conséquent pour H_m

$$H_m = -24\pi^2\bar{g}^{1/2}L_m = 192\pi^3 R_0^3 G^{3(4-3\gamma)} e^{3(\gamma-2)\bar{\Omega}} > 0 \qquad (1.63)$$

Nous écrirons symboliquement cette relation sous la forme:

$$H_m = \delta\lambda(\phi)e^{3(\gamma-2)\bar{\Omega}}$$

1.4.1 Equations de champs

L'Hamiltonien ADM correspondant au Lagrangien (1.62) s'écrit donc:

$$H^2 = p_+^2 + p_-^2 + 12\frac{p_\phi^2\phi^2}{3+2\omega} + 24\pi^2 R_0^6 e^{-6\bar{\Omega}}U + \delta\lambda e^{3(\gamma-2)\Omega}$$

On en déduit les équations de Hamilton:

$$\dot{\beta}_\pm = \frac{\partial H}{\partial p_\pm} = \frac{p_\pm}{H} \tag{1.64}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{12\phi^2 p_\phi}{(3+2\omega)H} \tag{1.65}$$

$$\dot{p}_\pm = -\frac{\partial H}{\partial \beta_\pm} = 0 \tag{1.66}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = -12\frac{\phi p_\phi^2}{(3+2\omega)H} + 12\frac{\omega_\phi \phi^2 p_\phi^2}{(3+2\omega)^2 H} - 12\pi^2 R_0^6 \frac{e^{-6\bar{\Omega}}U_\phi}{H} - \frac{\delta\lambda_\phi e^{3(\gamma-2)\Omega}}{2H} \tag{1.67}$$

$$\dot{H} = \frac{dH}{d\bar{\Omega}} = \frac{\partial H}{\partial \bar{\Omega}} = -72\pi^2 R_0^6 \frac{e^{-6\bar{\Omega}}U}{H} + 3/2\delta\lambda(\gamma-2)\frac{e^{3(\gamma-2)\Omega}}{H} \tag{1.68}$$

Les fonctions lapse et shift gardent la même forme que dans les sections précédentes et on utilise les mêmes fonctions x, y et z. En revanche la variable k représentant la présence de matière est désormais définie par

$$k^2 = \delta\lambda e^{3(\gamma-2)\Omega}H^{-2}$$

λ étant une fonction positive du champ scalaire, ou encore:

$$\begin{aligned} k^2 &= \delta\lambda x^\gamma y^{2-\gamma}U^{\gamma/2-1} \\ k^2 &= \delta\lambda x^2 e^{3(\gamma-2)\Omega} \\ k^2 &= \delta y^2 U^{-1}\lambda V^{-\gamma} \end{aligned} \tag{1.69}$$

Les équations de champs peuvent alors être réécrites comme:

$$\dot{x} = 72y^2x - 3/2(\gamma-2)k^2x \tag{1.70}$$

$$\dot{y} = y(6\ell z + 72y^2 - 3) - 3/2(\gamma-2)k^2y \tag{1.71}$$

$$\dot{z} = 24y^2(3z - \frac{\ell}{2}) - 3/2(\gamma-2)k^2z - 1/2\ell_m k^2 \tag{1.72}$$

où les quantités ℓ et ℓ_m sont définies par $\ell = \phi U_\phi U^{-1}(3+2\omega)^{-1/2}$ et $\ell_m = \phi\lambda_\phi\lambda^{-1}(3+2\omega)^{-1/2}$. Le couplage entre la matière et le champ scalaire fait donc apparaître un nouveau terme ℓ_m dans l'équation pour z. Quant à la contrainte Hamiltonienne, elle devient:

$$p^2x^2 + 24y^2 + 12z^2 + k^2 = 1 \tag{1.73}$$

L'équation pour le champ scalaire est à nouveau:

$$\dot{\phi} = 12z\frac{\phi}{(3+2\omega)^{1/2}}$$

1.4.2 Isotropisation lorsque $k \not\to 0$

Le seul point d'équilibre compatible avec une isotropisation de classe 1 est:

$$(0, \pm \frac{1}{4\sqrt{6}(\ell-\ell_m)}\left[4\ell_m(\ell_m-\ell)-3(\gamma-2)\gamma\right]^{1/2}, \frac{\gamma}{4(\ell-\ell_m)})$$

La contrainte Hamiltonienne impose alors que:

$$k^2 = \frac{2\ell(\ell-\ell_m)-3\gamma}{2(\ell-\ell_m)^2}$$

Lorsque $\ell_m = 0$, on retrouve évidement les points d'équilibre en l'absence de couplage entre la matière et le champ scalaire. La variable k est réelle tant que

$$\ell(\ell-\ell_m) > \frac{3}{2}\gamma$$

et les points d'équilibre sont réels et finis si:

$$4\ell_m(\ell_m-\ell) > 3(\gamma-2)\gamma$$

$$\ell \not\to \ell_m$$

c'est-à-dire $U \not\to \lambda$. Notons que la première condition est automatiquement satisfaite lorsque $\ell_m = 0$. De plus, comme ici $k \neq 0$, est fini et que nous étudions une isotropisation de classe 1 telle que $y \neq 0$, cela signifie que ℓ et ℓ_m ne peuvent pas diverger sauf ensemble au même ordre. En appliquant l'hypothèse de variabilité à la quantité $\frac{[2\ell_m+\ell(\gamma-2)]}{(\ell-\ell_m)}$ et en utilisant l'équation pour x, on calcule alors qu'à l'approche de l'équilibre isotrope

$$x \to x_0 e^{-\frac{3[2\ell_m+\ell(\gamma-2)]}{2(\ell-\ell_m)}\Omega}$$

où x_0 est une constante d'intégration. De même, les fonctions métriques tendront vers

$$e^{-\Omega} \to t^{\frac{2(\ell-\ell_m)}{3\ell\gamma}}$$

lorsque $\frac{3\ell\gamma}{2(\ell-\ell_m)}$ tend vers une constante non nulle. Or, ceci est toujours le cas puisque ℓ et ℓ_m ne peuvent pas diverger sauf ensemble au même ordre et que ℓ ne peut tendre vers zéro car alors k serait complexe. Le potentiel quant à lui tend vers t^{-2} et le champ scalaire se comporte asymptotiquement comme la solution de

$$\dot{\phi} = 3\gamma\left(\frac{U_\phi}{U} - \frac{\lambda_\phi}{\lambda}\right)^{-1}$$

Cette équation différentielle s'intègre facilement pour montrer que $U \to U_0\lambda V^{-\gamma}$ en accord avec le fait que k tende vers une constante non nulle. Comme $\lambda \propto UV^\gamma$ et $y \neq 0$, on déduit de la définition de y que

$$\lambda \to e^{\frac{3\gamma\ell_m\Omega}{\ell-\ell_m}}$$

et de la forme asymptotique des fonctions métriques que

$$\lambda \to t^{-2\frac{\ell_m}{\ell}}$$

Par conséquent de la condition de réalité de k, il vient que $\lambda > t^{-2(1-\frac{3}{2}\frac{\gamma}{\ell^2})}$.

1.4.3 Isotropisation lorsque $k \to 0$

On distingue deux cas selon que $\ell_m k^2 \to 0$ ou $\ell_m k^2 \not\to 0$.

$\ell_m k^2 \to 0$

Nous retrouvons les mêmes points d'équilibre et comportements asymptotiques que dans le cas du vide. Pour k^2 nous obtenons que $k^2 \to \lambda e^{2(3/2\gamma-\ell^2)\Omega}$ et les conditions $k \to 0$ et $\ell_m k \to 0$ se traduisent donc par les contraintes supplémentaires

$$\lambda e^{2(3/2\gamma-\ell^2)\Omega} \to 0$$

$$\ell_m \lambda e^{2(3/2\gamma-\ell^2)\Omega} \to 0$$

Lorsque ℓ_m ne diverge pas, cette deuxième condition est évidement automatiquement satisfaite lorsque la première l'est. De plus comme y ne tend pas vers zéro au contraire de k, nous avons $U >> \lambda V^{-\gamma}$, c'est-à-dire que le potentiel est supérieure à la densité d'énergie du fluide parfait.

$\underline{\ell_m k^2 \not\to 0}$

Comme $k \to 0$, cela signifie que ℓ_m doit diverger. Le point d'équilibre correspondant à la définition de la classe 1 est alors:

$$(x,y,z) = (0, \pm 1/(2\sqrt{6}),0)$$

avec $k^2 = -\ell\ell_m^{-1}$. Afin que k disparaisse et soit réel il faut donc respectivement que $\ell << \ell_m$ et $\ell\ell_m^{-1} < 0$. Il faut aussi que ℓ tende vers une constante non nulle ou diverge afin que $\ell_m k^2$ soit non nul. Dans ce dernier cas, z doit disparaître suffisamment vite afin que $z\ell$ reste fini. A l'approche du point d'équilibre, nous trouvons que $x \to e^{3\Omega}$, indiquant que l'Univers tend vers un modèle de De Sitter et le potentiel vers une constante. Comme précédemment, les contraintes $k \to 0$ et $\ell_m k^2 \not\to 0$, impliquent respectivement que

$$\lambda e^{3\gamma\Omega} \to 0$$

$$\ell_m \lambda e^{3\gamma\Omega} \not\to 0$$

Pour les mêmes raisons que plus haut, la limite $k \to 0$ implique que le potentiel est très supérieur à la densité d'énergie du fluide parfait. A partir de la valeur asymptotique de k et de sa définition (1.69), nous déduisons l'équation différentielle dont la solution correspond à la forme asymptotique pour ϕ:

$$\delta \frac{1}{\lambda_\phi} \frac{U_\phi}{U} = e^{3\gamma\Omega}$$

1.4.4 Discussion

Dans une première partie, nous résumons nos résultats et dans une seconde partie, nous les appliquons à des théories non minimalement couplées. L'univers peut s'isotropiser de 3 manières différentes selon que k tend vers une constante non nulle, nulle et tel que $\ell_m k^2 \to 0$ ou nulle et tel que $\ell_m k^2 \not\to 0$. Ci dessous, nous énonçons successivement les résultats obtenus pour chacune d'entre elles.

Résumé des résultats

Cas 1: Isotropisation avec $\Omega_m \not\to 0$

Soient les quantités $\ell = \phi U_\phi U^{-1}(3+2\omega)^{-1/2}$ et $\ell_m = \phi\lambda_\phi\lambda^{-1}(3+2\omega)^{-1/2}$. Des conditions nécessaires à l'isotropisation du modèle de Bianchi de type I en présence d'un champ scalaire massif minimalement couplé à la métrique mais non minimalement couplé au fluide parfait sont que

- *$\ell \not\to \ell_m$ (non divergence des points d'équilibre)*
- *$4\ell_m(\ell_m - \ell) > 3(\gamma - 2)\gamma$ (condition de réalité)*
- *$\ell(\ell - \ell_m) > \frac{3}{2}\gamma$ (condition de réalité)*
- *ℓ et ℓ_m restent finis ou divergent au même ordre (respect de la contrainte)*

A l'approche de l'isotropie, les fonctions métriques tendent vers une loi en puissance du temps propre $t^{\frac{2(\ell-\ell_m)}{3\ell\gamma}}$, $\lambda \to t^{-2\frac{\ell_m}{\ell}}$ tandis que le potentiel décroît comme t^{-2}. Le champ scalaire vérifie asymptotiquement que $U \to U_0\lambda e^{3\gamma\Omega}$.

Cette dernière relation permet de déterminer la forme asymptotique de ϕ et donc celles de ℓ et ℓ_m. Il est interessant de noter que jusque là, le fait que $\Omega_m \not\to 0$ aboutissait toujours à la convergence des fonctions métriques vers la fonction $t^{\frac{2}{3\gamma}}$, interdisant une accélération tardive de notre Univers. On voit que le couplage λ entre le champ scalaire et le fluide parfait permet d'introduire cette possibilité pour un champ minimalement couplé. Il serait ainsi possible de résoudre le problème de coïncidence qui résulte dans le fait que les paramètres de densité du fluide parfait et de l'énergie sombre soient aujourd'hui du même ordre.

Cas 2: Isotropisation avec $\Omega_m \to 0$ et $\Omega_m\ell_m \to 0$

Soient les quantités $\ell = \phi U_\phi U^{-1}(3+2\omega)^{-1/2}$ et $\ell_m = \phi\lambda_\phi\lambda^{-1}(3+2\omega)^{-1/2}$. Les conditions nécessaires à l'isotropisation sont:

- *$\ell^2 < 3$ (condition de réalité)*
- *$\lambda e^{2(3/2\gamma - \ell^2)\Omega} \to 0$ (condition pour que $k \to 0$)*

– $\ell_m \lambda e^{2(3/2\gamma-\ell^2)\Omega} \to 0$ *(condition pour que* $\ell_m k^2 \to 0$*)*

Si ℓ^2 tend vers une constante non nulle, les fonctions métriques tendent vers $t^{\ell^{-2}}$ et le potentiel décroît comme t^{-2}. Si ℓ^2 tend vers zéro, l'Univers tend vers un modèle de De Sitter et le potentiel vers une constante. Le comportement asymptotique du champ scalaire est solution de l'équation $\dot{\phi} = 2\frac{\phi^2 U_\phi}{U(3+2\omega)}$.

Pour des raisons de clarté, nous avons choisi d'exprimer les limites $\Omega_m \to 0$ et $\Omega_m \ell_m \to 0$ ci-dessus (et ci-dessous) en fonction de $e^{-\Omega}$ et de ϕ, ces 2 quantités étant définies dans ce résultat par le comportement asymptotique des fonctions métriques et du champ scalaire.

Cas 3: Isotropisation avec $\Omega_m \to 0$ et $\Omega_m \ell_m \not\to 0$

Soient les quantités $\ell = \phi U_\phi U^{-1}(3+2\omega)^{-1/2}$ et $\ell_m = \phi \lambda_\phi \lambda^{-1}(3+2\omega)^{-1/2}$. Les conditions nécessaires à l'isotropisation sont que

– ℓ_m *diverge et* $\ell \to const \neq 0$ *ou diverge mais tel que* $z\ell \to 0$ *(condition pour que* $\ell_m k^2 \to 0$*).*
– $\ell << \ell_m$ *ou* $\lambda e^{3\gamma\Omega} \to 0$*(condition pour que* $k \to 0$*)*
– $\ell \ell_m^{-1} < 0$ *(condition de réalité)*

L'Univers tend vers un modèle de De Sitter et le potentiel vers une constante. Le champ scalaire vérifie asymptotiquement l'équation $\delta \frac{1}{\lambda_\phi}\frac{U_\phi}{U} = e^{3\gamma\Omega}$.

Applications aux théories non minimalement couplées

Dans ce qui suit, nous étudions 4 classes de théories minimalement couplées auxquelles appartiennent, après une transformation conforme, les théories de Brans-Dicke et des cordes lorsque le potentiel a une forme en puissance ou en exponentielle du champ scalaire. Nous rappelons la transformation conforme permettant de passer du référentiel de Brans-Dicke au référentiel d'Einstein et donc de la théorie non-minimalement couplée à la théorie minimalement couplée:

$$g_{\alpha\beta} = G\bar{g}_{\alpha\beta} = \lambda^{[3(4-3\gamma)]^{-1}}\bar{g}_{\alpha\beta}$$

G étant la fonction de gravitation de la théorie non minimalement couplée. C'est dans le référentiel d'Einstein que les résultats que nous venons d'énoncer trouvent leur place mais les conditions nécessaires à l'isotropie sont invariantes par la transformation conforme ci-dessus. En effet, si les fonctions métriques du référentiel d'Einstein tendent toutes vers une même fonction, la transformation conforme ci-dessus ne change pas cet état de fait.
Nous illustrerons chacune des applications avec des figures montrant les comportements de x, y, z, k, ϕ et ℓ dans le référentiel d'Einstein et dans le temps Ω avec les conditions initiales $\phi_0 = 0.14$, $y_0 = 0.25$, $z_0 = 0.12$. x_0 est calculé en utilisant la contrainte (1.73) avec $p_+^2 + p_-^2 = p^2 = 1$, $R_0^3 = 1/(2\sqrt{6}\pi)$ et $\delta = 1$. Les comportements des fonctions métriques dans le référentiel de Brans-Dicke seront montrés dans le temps propre avec les conditions initiales $\alpha_0 = -1.53$, $\beta_0 = -1.25$, $\gamma_0 = 0.12$, $d\alpha_0/d\tau_0 = 2.48$, $d\beta_0/d\tau_0 = 1.55$ et $d\gamma/d\tau_0 = 0.33$, le temps τ étant défini par $dt = e^{-3\Omega}d\tau$.

Théories de Brans-Dicke avec un potentiel en exponentiel du champ scalaire

Considérons la classe de théorie définie par (1.62) et telle que:

$$\begin{aligned}
\omega &= \omega_0 \\
U &= \phi^{-2}e^{n\phi} \\
\lambda &= \phi^m
\end{aligned}$$

La transformation conforme montre que cette théorie correspond à une théorie non minimalement couplée définie par (1.60) avec:

$$\begin{aligned}
G &= \phi^{\frac{m}{3(4-3\gamma)}} \\
\omega &= \left[\frac{3}{2}(1 - \frac{m^2}{9(4-3\gamma)^2}) + \omega_0)\right]\phi^{\frac{-m}{3(4-3\gamma)}-1} \\
U &= \phi^{-2(1+\frac{m}{3(4-3\gamma)})}e^{n\phi}
\end{aligned}$$

La théorie de Brans-Dicke avec un potentiel exponentiel correspond alors à $m = 3(3\gamma - 4)$. Les quantités ℓ et ℓ_m sont définies par:

$$\ell = \frac{n\phi - 2}{\sqrt{3 + 2\omega_0}}$$

$$\ell_m = \frac{m}{\sqrt{3 + 2\omega_0}}$$

Il s'ensuit que $3 + 2\omega_0$ doit être positif. ℓ_m ne peut pas diverger et par conséquent le cas 3 ne se produit pas.

Pour le cas 1, à l'approche de l'état d'équilibre isotrope le champ scalaire se comporte comme:

$$e^{n\phi}\phi^{-(2+m)} \to U_0 e^{3\gamma\Omega}$$

Comme ℓ est fini, ϕ ne peut pas diverger et devrait disparaître asymptotiquement, impliquant que $m < -2$ et finalement que $\phi \to e^{\frac{3\gamma}{-(2+m)}\Omega}$. La seconde condition de réalité s'écrit alors:

$$\frac{4(2 + m) - 3\gamma(3 + 2\omega_0)}{2(3 + 2\omega_0)} > 0$$

Mais $m < -2$, $\gamma > 0$ et $3 + 2\omega_0 > 0$ et donc cette condition ne peut être satisfaite. Par conséquent, une isotropisation de classe 1 ne se produit pas.

Considérons à présent le cas 2. Intégrant l'équation différentielle pour ϕ, nous obtenons:

$$\phi = \frac{2}{n - \phi_0 e^{\frac{4\Omega}{3+2\omega_0}}}$$

Alors, lorsque $\Omega \to -\infty$, $\phi \to 2n^{-1}$, $\ell \to 0$ et λ tend vers la constante $(2n^{-1})^m$. Si l'Univers s'isotropise, il tendra vers un modèle de De Sitter. Remarquons que ϕ et donc n doivent être positifs afin que λ soit une fonction réelle.

Utilisant la transformation conforme, lorsque l'isotropisation se produit dans le référentiel de Brans-Dicke ou ϕ est non minimalement couplé à la courbure et puisque λ tend vers une constante, l'Univers tend également vers un modèle de De Sitter. L'évolution des variables et des fonctions α, β et γ ainsi que leur dérivées par rapport au temps propre est illustré par la figure 1.9.
Une isotropisation de classe 2 est aussi possible lorsque $n < 0$ et est tracé sur la figure 1.10. Comme noté

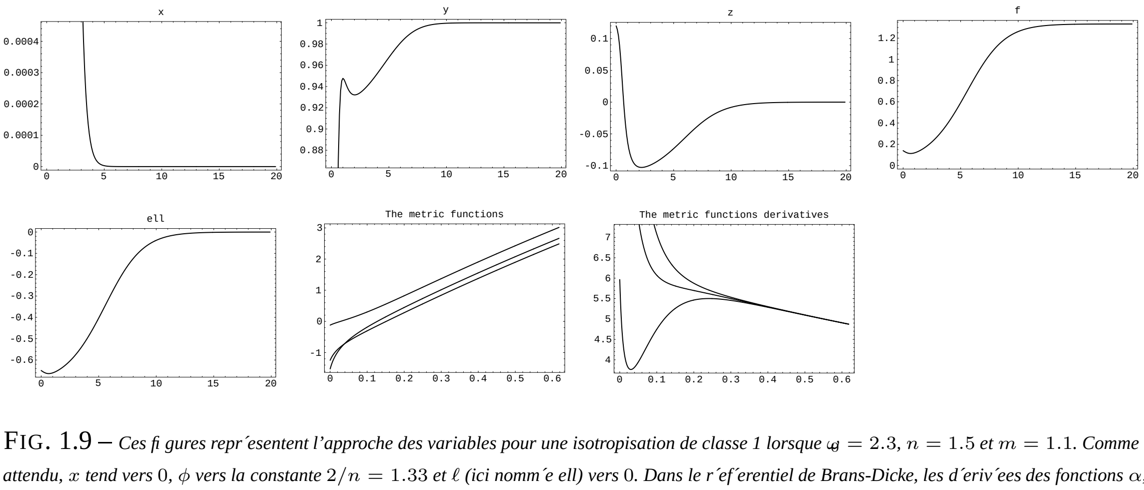

FIG. 1.9 – *Ces figures représentent l'approche des variables pour une isotropisation de classe 1 lorsque* $\omega = 2.3$, $n = 1.5$ *et* $m = 1.1$. *Comme attendu,* x *tend vers* 0, ϕ *vers la constante* $2/n = 1.33$ *et* ℓ *(ici nommé ell) vers* 0. *Dans le référentiel de Brans-Dicke, les dérivées des fonctions* α, β *et* γ *tendent vers une fonction commune, montrant l'isotropisation.*

ci-dessus, un tel intervalle pour n est impossible pour une isotropisation de classe 1 car λ serait complexe.

Théories de Brans-Dicke avec un potentiel en puissance du champ scalaire

Considérons la classe de théorie définie par (1.62) et telle que:

$$\omega = \omega_0$$

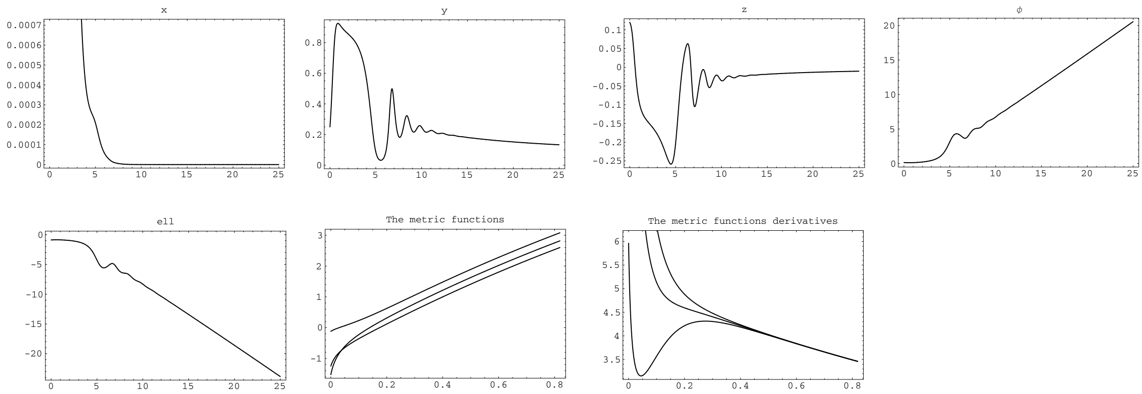

FIG. 1.10 – *Ces figures repr´esentent l'approche des variables pour une isotropisation de classe 2 lorsque $\omega = 2.3$, $n = -3.1$ et $m = 1.1$. x tend toujours vers z´ero mais y aussi. ϕ et donc ℓ diverge. Notons que ϕ, y, z et ℓ subissent des oscillations amorties. Dans le r´ef´erentiel de Brans-Dicke, les d´eriv´ees des fonctions m´etriques α, β et γ tendent vers une fonction commune, montrant l'isotropisation.*

$$U = \phi^n$$
$$\lambda = \phi^m$$

Dans le référentiel de Brans-Dicke cette théorie correspond à la théorie tenseur-scalaire non minimalement couplée définie par:

$$G = \phi^{\frac{m}{3(4-3\gamma)}} \quad (1.74)$$
$$\omega = \left[\frac{3}{2}(1-\frac{m^2}{9(4-3\gamma)^2})+\omega_0)\right]\phi^{\frac{-m}{3(4-3\gamma)}-1} \quad (1.75)$$
$$U = \phi^{n-\frac{2m}{3(4-3\gamma)}} \quad (1.76)$$
$$(1.77)$$

La théorie de Brans-Dicke avec un potentiel en puissance de ϕ est obtenue pour $m = 3(3\gamma-4)$. On calcule que:

$$\ell = \frac{n}{\sqrt{3+2\omega_0}}$$
$$\ell_m = \frac{m}{\sqrt{3+2\omega_0}}$$

avec $3+2\omega_0 > 0$. A nouveau ℓ_m ne peut pas diverger et le cas 3 est exclu.

Pour le cas 1, il est nécessaire que $n \neq m$ afin que $\ell \not\to \ell_m$. Asymptotiquement, le champ scalaire se comporte comme:

$$\phi \to \phi_0 e^{-\frac{3\gamma}{m-n}\Omega}$$

Par conséquent, en $\Omega \to -\infty$, $\phi \to 0$(ϕ diverges) si $m-n < 0$ (respectivement $m-n > 0$). Les conditions de réalités s'écrivent:

$$4m(m-n)+3\gamma(2-\gamma)(3+2\omega_0) > 0$$
$$2n(n-m)-3\gamma(3+2\omega_0) > 0$$

La seconde sera respectée si $n > 0 (n < 0)$ lorsque $\phi \to 0$(respectivement lorsque ϕ diverge). Nous trouvons qu'à l'approche de l'isotropie, les fonctions métriques tendent vers $t^{\frac{2(n-m)}{3n\gamma}}$ et $\lambda \to t^{-\frac{2m}{n}}$.

Utilisant la transformation conforme, nous déduisons pour la théorie non minimalement couplée que les fonctions métriques tendront vers:

$$t^{\frac{m(8-5\gamma)+2n(3\gamma-4)}{\gamma[m+3n(3\gamma-4)]}}$$

Tous ces comportements sont illustrés sur la figure 1.11.

Pour le cas 2, nous obtenons pour ϕ:

$$\phi \to e^{\frac{2n}{3+2\omega_0}\Omega}$$

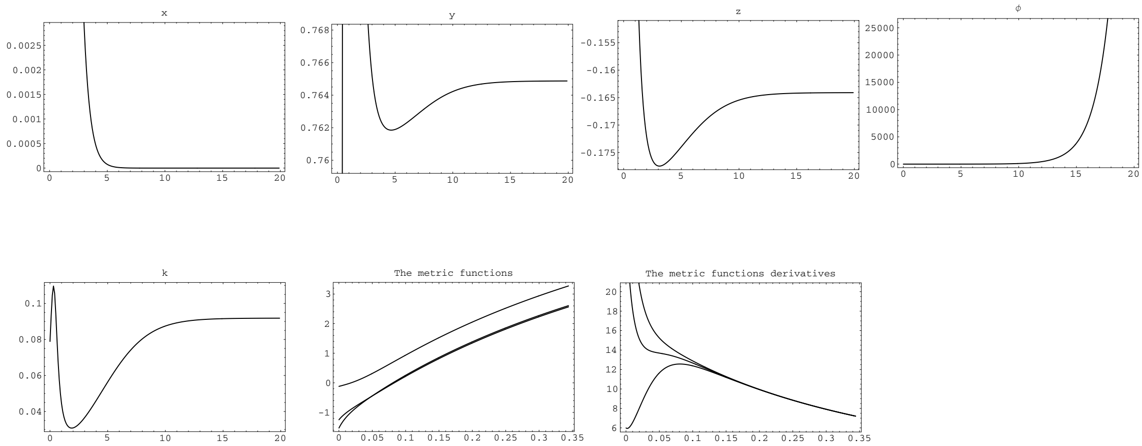

FIG. 1.11 – *Ces figures représentent l'approche des variables pour une isotropisation de classe 1 lorsque $\omega_0 = 2.3$, $n = -3.1$ et $m = 1.1$. A nouveau x disparaît, y et k tendent vers des constantes non nulles montrant que $U \propto \lambda e^{3\gamma\Omega}$. ϕ diverge car $m - n > 0$. Dans le référentiel de Brans-Dicke, l'Univers s'isotropise.*

Ainsi k tendra vers zéro lorsque $\Omega \to -\infty$ si $2n(m-n) + 3\gamma(3+2\omega_0) > 0$ et la condition de réalité pour les points d'équilibre sera respectée si $n^2(3+2\omega_0)^{-1} < 3$. Les fonctions métriques tendent alors vers $t^{(3+2\omega_0)n^{-2}}$ lorsque $n \neq 0$ ou vers un modèle de De Sitter lorsque $n = 0$.

Dans le référentiel de Brans-Dicke ou le champ scalaire est non minimalement couplé à la courbure, les fonctions métriques tendront vers:

$$t^{\frac{mn+3(3\gamma-4)(3+2\omega_0)}{n[m+3n(3\gamma-4)]}}$$

lorsque $n \neq 0$. Si $n = 0$, le comportement des fonctions métriques est le même que dans le référentiel d'Einstein et l'Univers tend vers un modèle de De Sitter. Ce cas est illustré par la figure 1.12

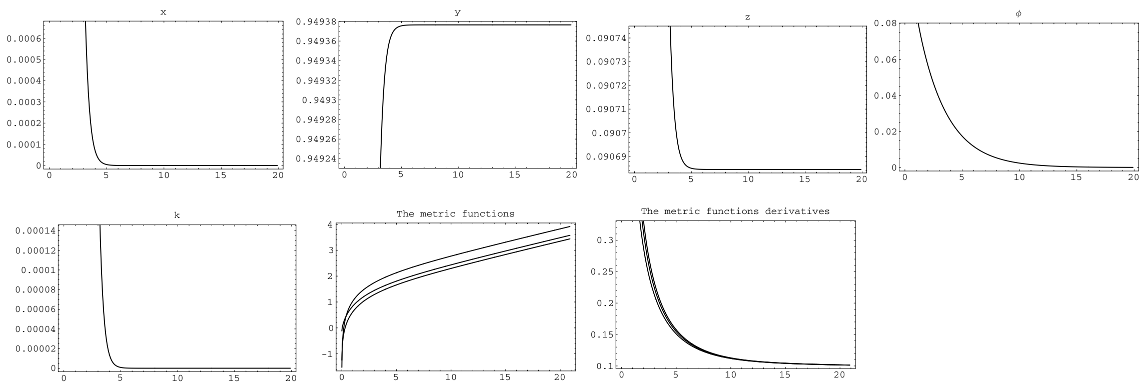

FIG. 1.12 – *Ces figures représentent l'approche des variables pour une isotropisation de classe 1 lorsque $\omega_0 = 2.3$, $n = 1.5$ et $m = 1.1$. Ici, k tend vers zéro.*

Théorie des cordes à basse énergie avec un potentiel en exponentiel du champ scalaire

Nous considérons la théorie définie par (4) et telle que:

$$\begin{aligned} \omega &= \omega_0\phi^2 + \omega_1 \\ U &= e^{n\phi} \\ \lambda &= e^{m\phi} \end{aligned}$$

Elle correspond à la théorie non minimalement couplée suivante:

$$G = e^{\frac{m}{3(4-3\gamma)}\phi}$$

$$\omega = \left[\frac{\frac{3}{2}+\omega_0\phi^2+\omega_1}{\phi^2} - \frac{3m^2}{18(4-3\gamma)^2}\right]\phi e^{\frac{-m}{3(4-3\gamma)}\phi}$$
$$U = e^{(n-\frac{2m}{3(4-3\gamma)})\phi}$$

La théorie des cordes à basse énergie avec un potentiel en exponentiel du champ scalaire est retrouvée pour $m = 3(4-3\gamma)$, $\omega_0 = 5/2$ et $\omega_1 = -3/2$. Nous calculons que:

$$\ell = \frac{n\phi}{\sqrt{3+2\phi^2\omega_0+2\omega_1}}$$

$$\ell_m = \frac{m\phi}{\sqrt{3+2\phi^2\omega_0+2\omega_1}}$$

Ces expressions montrent que nous n'aurons jamais $\ell << \ell_m$ et donc le cas 3 ne se produit pas.

En ce qui concerne le cas 1, il est nécessaire que $m \neq n$. De plus, nous trouvons pour le champ scalaire:

$$\phi = \phi_0 + \frac{3\gamma\Omega}{n-m}$$

Ainsi, ϕ diverge et ℓ et ℓ_m tendent vers des constantes qui seront réelles si $\omega_0 > 0$. Les conditions de réalité s'écrivent:

$$2m(m-n)+3(2-\gamma) > 0$$

$$n(n-m)-3\gamma\omega_0 > 0$$

ω_0 étant positif, la seconde condition nécessite $n(n-m) > 0$ et donc $n \neq 0$. Par conséquent, lorsque l'isotropisation se produit, les fonctions métriques et λ tendent respectivement vers $t^{2\frac{n-m}{3n\gamma}}$ et $t^{-2\frac{m}{n}}$.

Nous déduisons que dans le référentiel de Brans-Dicke, lorsque l'isotropisation se produit, les fonctions métriques tendent vers:

$$t^{\frac{m(8-5\gamma)+2n(3\gamma-4)}{\gamma[m+3n(3\gamma-4)]}}$$

Ce cas est représenté par la figure 1.13.

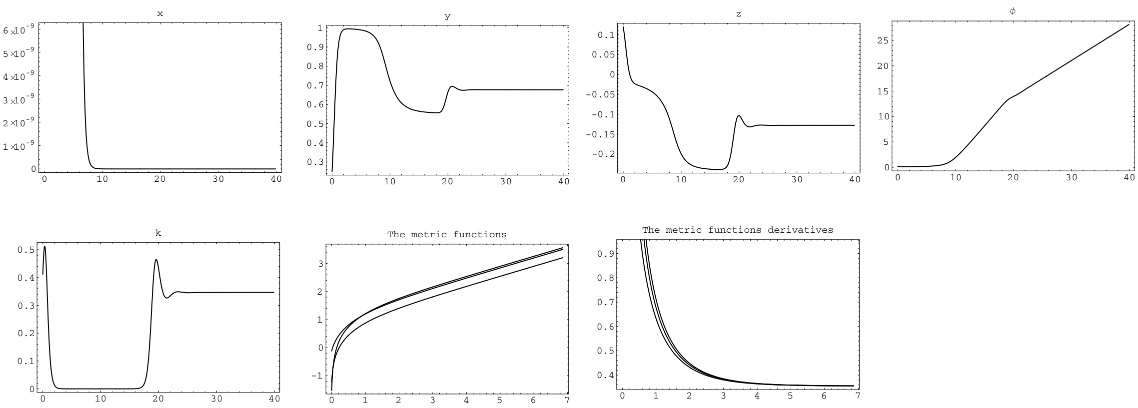

FIG. 1.13 – *Ces figures représentent l'approche des variables pour une isotropisation de classe 1 lorsque $\omega_0 = 2.3$, $n = -3.1$ et $m = 1.1$. k tend vers une constante mais nous remarquons également l'existence, avant l'équilibre, d'une période durant laquelle le paramètre de densité du fluide parfait est quasiment nul.*

En ce qui concerne le cas 2, le champ scalaire se comporte asymptotiquement comme:

$$\phi = \frac{2n(\Omega-\phi_0) \pm \sqrt{8\omega_0(3+2\omega_1)+4n^2(\phi_0-\Omega)^2}}{4\omega_0}$$

Par conséquent, en fonction du signe de la racine carré, nous avons deux branches telles que $\phi \to 0$ ou $\phi \to n\omega_0^{-1}\Omega$.
Pour la première, $\ell \to 0$ et l'Univers tend vers un modèle de De Sitter. La limite permettant la disparition

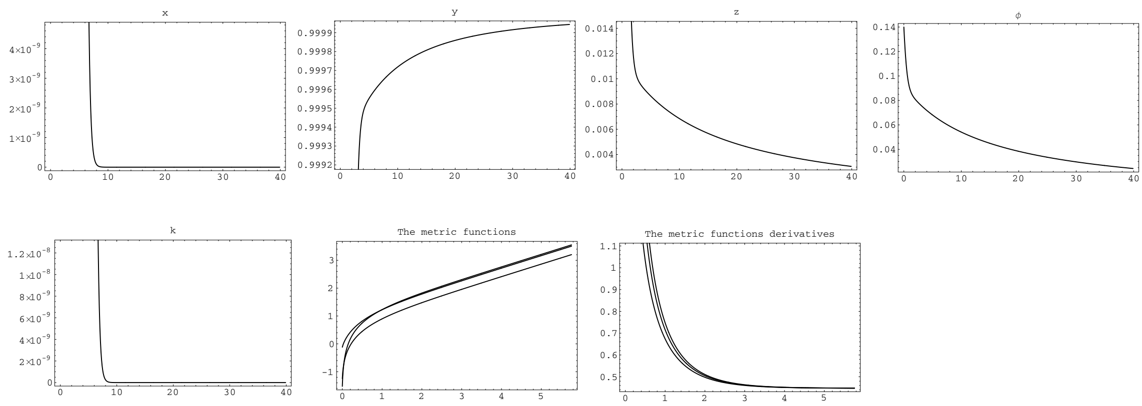

FIG. 1.14 – *Ces figures représentent l'approche des variables pour une isotropisation de classe 1 lorsque* $\omega_0 = 2.3$, $\omega_1 = 0.5$, $n = 1.5$ *et* $m = 1.1$. k *et* ϕ *tendent vers zéro.*

de k est toujours respectée. Une simulation numérique de ce cas est représentée par la figure 1.14.
Pour la seconde, $\ell \to n(2\omega_0)^{-1/2}$ et ainsi l'isotropisation nécessite $\omega_0 > 0$ et $n^2(2\omega_0)^{-1} < 3$. Si $n \neq 0$, les fonctions métriques tendent vers $t^{\frac{2\omega_0}{n^2}}$ et la limite permettant la disparition de k est satisfaite si $\ell^2 < \frac{3\gamma}{2}$. Si $n = 0$, l'Univers tend vers un modèle de De Sitter et la limite sur k est toujours satisfaite.

A nouveau, dans le référentiel de Brans-Dicke, nous déduisons que lorsque l'isotropisation se produit et le champ scalaire tend vers zéro ou $n = 0$, les fonctions métriques tendent vers la même forme que dans le référentiel d'Einstein car λ tend vers une constante. Lorsque le champ scalaire diverge et $n \neq 0$, elles tendent vers:

$$t^{\frac{n^2(9\gamma-13)+3(7\gamma-8)\omega_0}{n^2(9\gamma-13)+3\gamma\omega_0}}$$

Théorie des cordes à basse énergie avec un potentiel en puissance du champ scalaire

Nous considérons maintenant le Lagrangien minimalement couplé défini par:

$$\begin{aligned}\omega &= \omega_0\phi^2 + \omega_1\\ U &= \phi^p e^{n\phi}\\ \lambda &= e^{m\phi}\end{aligned}$$

et correspondant à la théorie non minimalement couplée suivante:

$$\begin{aligned}G &= e^{\frac{m}{3(4-3\gamma)}\phi}\\ \omega &= \left[\frac{\frac{3}{2} + \omega_0\phi^2 + \omega_1}{\phi^2} - \frac{3m^2}{18(4-3\gamma)^2}\right]\phi e^{\frac{-m}{3(4-3\gamma)}\phi}\\ U &= \phi^p e^{(n - \frac{2m}{3(4-3\gamma)})\phi}\end{aligned}$$

On obtient la théorie des cordes à basse énergie avec un potentiel en puissance du champ scalaire lorsque $m = 3(4-3\gamma)$, $n = 2$, $\omega_0 = 5/2$ et $\omega_1 = -3/2$. Nous calculons que:

$$\ell = \frac{p + n\phi}{\sqrt{3 + 2\phi^2\omega_0 + 2\omega_1}}$$

$$\ell_m = \frac{m\phi}{\sqrt{3 + 2\phi^2\omega_0 + 2\omega_1}}$$

Encore une fois, il est impossible que ℓ_m diverge et $\ell << \ell_m$ et donc le cas 3 est exclu.

Pour le cas 1, nous trouvons que le champ scalaire se comporte comme:

$$\phi = p(m-n)^{-1} ProductLog((n-m)e^{3\gamma p^{-1}(\Omega - \phi_0)})$$

Lorsque $p\gamma^{-1} > 0$, le champ scalaire disparaît, sinon il diverge. Alors, $(n-m)p^{-1}$ doit être positif sinon le potentiel est complexe.
Lorsque $\phi \to 0$, il est nécessaire que $3+2\omega_1 > 0$ tel que ℓ et ℓ_m soit réel et les conditions de réalité pour les points d'équilibre se réduisent à $2p^2 - 3\gamma(3+2\omega_1) > 0$. Alors les fonctions métriques tendent vers $t^{\frac{2}{3\gamma}}$ et λ vers une constante. Ce cas est illustré sur la figure 1.15.
Lorsque $\phi \to \infty$, il est nécessaire que $\omega_0 > 0$ tel que ℓ et ℓ_m soit réel et $n \neq m$ tel que ℓ ne tende pas

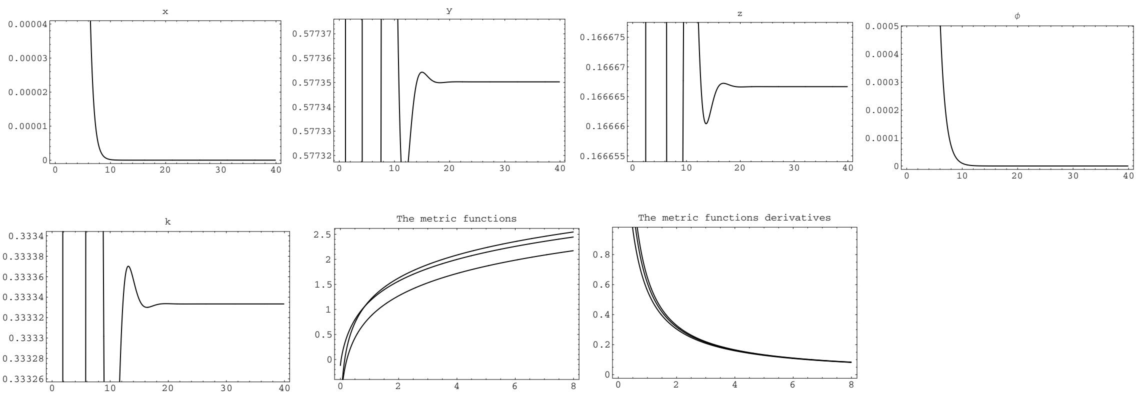

FIG. 1.15 – *Ces figures représentent l'approche des variables pour une isotropisation de classe 1 lorsque* $\omega_0 = 2.3$, $\omega_1 = 0.5$, $n = -3.1$, $m = 1.1$ *et* $p = 3$. k *oscille vers une constante et* ϕ *tend vers zéro. Notons les fortes oscillations de* y, z *et* k.

vers ℓ_m. Les conditions de réalité des points d'équilibre s'écrivent alors $2m(m-n) + 3\gamma(2-\gamma)\omega_0 > 0$ et $n(n-m) - 3\gamma\omega_0 > 0$, impliquant que $n(n-m) > 0$ et $n \neq 0$. Les fonctions métriques tendent vers $t^{\frac{2(n-m)}{3n\gamma}}$ et $\lambda \to t^{-2\frac{m}{n}}$. Des figures similaires aux figures 1.15 mais avec un champ scalaire divergeant peuvent être obtenues.

Dans le référentiel de Brans-Dicke, les fonctions métriques tendent vers la même forme que dans le référentiel d'Einstein durant l'isotropisation si $\phi \to 0$. Lorsque ϕ diverge, elles tendent vers:

$$t^{\frac{m(8-5\gamma)+2n(3\gamma-4)}{\gamma[m+3n(3\gamma-4)]}}$$

Examinons le cas 2. Le champ scalaire est tel que:

$$\phi_0 + 1/2\left[\frac{(3+2\omega_1)\ln\phi}{p} - \frac{n^2(3+2\omega_1)+2p^2\omega_0}{pn^2}\ln(p+n\phi) + \frac{2\omega_0\phi}{n}\right] = \Omega$$

Ainsi, il existe trois comportements possibles du champ scalaire tel que $\Omega \to -\infty$.

Le premier est tel que ϕ tende vers zéro et il est alors nécessaire que $p > 0$ et $3+2\omega_1 > 0$. On calcule que $\ell \to p(3+2\omega_1)^{-1/2}$ impliquant $p^2(3+2\omega_1)^{-1} < 3$. Les fonctions métriques tendent vers $t^{(3+2\omega_1)/p^2}$ et k tend toujours vers 0 tant que $\ell^2 < 3/2\gamma$. Ce cas est montré sur la figure 1.16. Puisque ϕ tend vers zéro, λ tend vers une constante et les résultats sont identiques dans le référentiel de Brans-Dicke.
Le second est tel que ϕ diverge comme $\frac{n}{2\omega_0}\Omega$. Il doit être positif et les expressions de ℓ et ℓ_m seront alors réelles si $\omega_0 > 0$. Ceci implique que la divergence positive de ϕ nécessite $n < 0$. Alors, ℓ tend vers $n(2\omega_0)^{-1/2}$ et il vient qu'une condition nécessaire à l'isotropisation est $n^2(2\omega_0)^{-1} > 3$. Les fonctions métriques tendent vers $t^{\frac{2\omega_0}{n^2}}$ et k vers 0 si $n(m-2n) + 6\gamma\omega_0 > 0$. Dans le référentiel de Brans-Dicke, nous trouvons que les fonctions métriques tendent vers $t^{\frac{mn+12\omega_0(3\gamma-4)}{mn}}$.
Enfin, le troisième comportement du champ scalaire est tel que $\phi \to -pn^{-1}$ et Ω diverge négativement si $\left[-n^2(3+2\omega_1) - 2k^2\omega_0\right](pn^2)^{-1} > 0$. Alors, $\ell \to 0$ et l'Univers tend vers un modèle de De Sitter. La condition $k \to 0$ est toujours respectée. Une fois de plus, λ tend vers une constante et, dans le référentiel de Brans-Dicke, les fonctions métriques tendent vers la même forme que dans le référentiel d'Einstein.

Ceci termine le chapitre sur l'isotropisation du modèle de Bianchi de type I. Dans le chapitre suivant, nous allons voir comment traiter les modèles avec courbure.

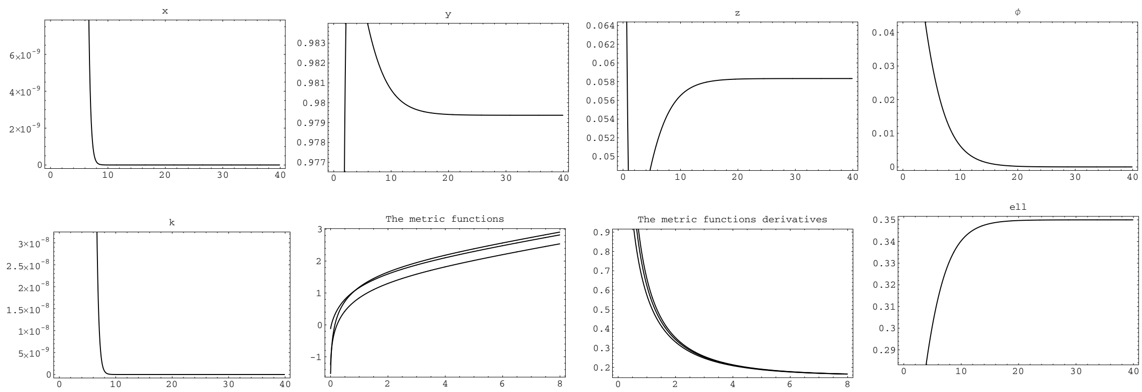

FIG. 1.16 – *Ces fi gures repr´esentent l'approche des variables pour une isotropisation de classe 1 lorsque* $\omega = 2.3$, $\omega_1 = 0.5$, $n = -3.1$, $m = 1.1$ *et* $p = 0.7$. k *et* ϕ *tendent vers z´ero.* ℓ *tend vers* 0.35 *ce qui est plus petit que* $3/2\gamma = 3/2$

Chapitre 2

Les modèles de Bianchi avec courbure(2 articles)

Dans les deux sections suivantes, nous allons considérer la présence de courbure en étudiant le processus d'isotropisation des modèles de Bianchi de la classe A, c'est-à-dire de type II, VI_0, VII_0, $VIII$ et IX. Ce dernier modèle en particulier, contient les solutions des modèles FLRW à courbure positive. Comme pour le modèle de Bianchi de type I, nous commencerons par examiner ce qui se passe sans, puis avec une fluide parfait.

2.1 Equations de champs

L'hamiltonien ADM pour les modèles avec courbure s'écrit:

$$H^2 = p_+^2 + p_-^2 + 12\frac{p_\phi^2\phi^2}{3+2\omega} + 24\pi^2 R_0^6 e^{-6\Omega}U + \delta e^{3(\gamma-2)\Omega} + V(\Omega,\beta_+,\beta_-) \tag{2.1}$$

où $V(\Omega,\beta_+,\beta_-)$ est le potentiel de courbure caractérisant chaque modèle de Bianchi et figurant dans le tableau 2.1. Les équations de Hamilton sont alors:

$$\dot{\beta}_\pm = \frac{\partial H}{\partial p_\pm} = \frac{p_\pm}{H} \tag{2.2}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{12\phi^2 p_\phi}{(3+2\omega)H} \tag{2.3}$$

$$\dot{p}_\pm = -\frac{\partial H}{\partial \beta_\pm} = -\frac{\partial V}{2H\partial\beta_\pm} \tag{2.4}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = -12\frac{\phi p_\phi^2}{(3+2\omega)H} + 12\frac{\omega_\phi \phi^2 p_\phi^2}{(3+2\omega)^2 H} - 12\pi^2 R_0^6 \frac{e^{-6\Omega}U_\phi}{H} \tag{2.5}$$

$$\dot{H} = \frac{dH}{d\Omega} = \frac{\partial H}{\partial \Omega} = -72\pi^2 R_0^6\frac{e^{-6\Omega}U}{H} + 3/2\delta(\gamma-2)\frac{e^{3(\gamma-2)\Omega}}{H} + \frac{\partial V}{2H\partial\Omega} \tag{2.6}$$

La définition d'un état isotrope reste inchangée par rapport à celle du modèle de Bianchi de type I. Mais désormais les moments conjugués des fonctions $\beta_\pm$ ne sont plus des constantes et donc lorsque l'on écrit qu'une condition nécessaire à l'isotropie est $d\beta_\pm/dt \to 0$, celle ci se traduit par $p_\pm e^{3\Omega} \to 0$.

Bianchi type	$V(\Omega,\beta_+,\beta_-)$
II	$12\pi^2 R_0^4 e^{4(-\Omega+\beta_+ + \sqrt{3}\beta_-)}$
VI_0, VII_0	$24\pi^2 R_0^4 e^{-4\Omega+4\beta_+}(\cosh 4\sqrt{3}\beta_- \pm 1)$
$VIII$, IX	$24\pi^2 R_0^4 e^{-4\Omega}[e^{4\beta_+}(\cosh 4\sqrt{3}\beta_- - 1)+$
	$1/2e^{-8\beta_+} \pm 2e^{-2\beta_+}\cosh 2\sqrt{3}\beta_-]$

TAB. 2.1 – *Potentiel de courbure des modèles de Bianchi de la classe A.*

Contrairement au modèle de Bianchi de type I, cette limite ne nous assure plus que l'isotropie se produit lorsque l'Univers est en expansion infinie en $\Omega \to -\infty$. Supposons que l'isotropisation de l'Univers conduise à un Univers statique, c'est-à-dire tel que $\Omega \to const$, lorsque le temps propre diverge. Alors, afin que $d\beta_\pm/dt$ disparaissent, il faut que $p_\pm \to 0$. Cependant les équations (2.4) indique que $dp_\pm/dt \propto -\frac{\partial V}{\partial \beta_\pm}e^{3\Omega}$. Par conséquent si Ω et $\beta_\pm$ tendent vers des constantes lorsque $t \to +\infty$, pour les modèles de Bianchi de type II, VI_0 et $VIII$, ces dérivées tendent vers des constantes non nulles et les moments conjugués $p_\pm$ ne peuvent pas disparaître et l'isotropie se produire. En revanche, les choses ne sont pas si simples pour les modèles de Bianchi de type VII_0 et IX car si $\beta_\pm \to 0$, il en est de même de $\frac{\partial V}{\partial \beta_\pm}$ et on ne peut rien dire sur les valeurs asymptotiques de $p_\pm$. Nous montrerons plus loin que pour ces modèles également, l'isotropie ne peut surgir que pour une valeur divergente de Ω.
Par conséquent, pour les modèles de Bianchi avec courbure, l'isotropisation ne peut se produire que si:

$$\Omega \to \pm\infty$$
$$\frac{d\beta_\pm}{d\Omega} \to 0$$
$$p_\pm e^{3\Omega} \to 0$$

Dans ce qui suit, l'hypothèse de variabilité de ℓ^2 sera systématiquement appliquée. Nous n'avons pas exploré ce qui se passe lorsque celle ci est levée. En effet, nous verrons que les résultats obtenus pour l'isotropisation des modèles avec courbure sont similaires à ceux obtenus pour un modèle plat. En revanche les conditions à vérifier pour montrer que l'isotropie est atteinte sont bien plus nombreuses et l'hypothèse de variabilité ne peut être levée facilement sans alourdir les calculs. Notons cependant que ceci est techniquement faisable comme montré pour le modèle de Bianchi de type I.
Afin de décrire la courbure des modèles de Bianchi nous introduirons de nouvelles variables préfixées w, similaires aux trois variables N_i ($i = 1,2,3$) définies par des arguments de symétrie des constantes de structure dans [126] et [25].

2.2 Dans le vide

Les résultats qui suivent ont été publiés dans [127], reproduit en annexe.

2.2.1 Modèle de Bianchi de type II

Afin de réécrire les équations de champs, nous utilisons les variables suivantes:

$$x_\pm = p_\pm H^{-1} \tag{2.7}$$
$$y = \pi R_0^3 \sqrt{U} e^{-3\Omega} H^{-1} \tag{2.8}$$
$$z = p_\phi \phi (3+2\omega)^{-1/2} H^{-1} \tag{2.9}$$
$$w = \pi R_0^2 e^{-2\Omega + 2(\beta_+ + \sqrt{3}\beta_-)} H^{-1} \tag{2.10}$$

Une seule variable w suffit à décrire la courbure de ce modèle de même qu'une seule variable N_i était suffisante dans [25]. Alors la condition $\frac{d\beta_\pm}{d\Omega} \to 0$ nécessaire à l'isotropisation se traduit par $x_\pm \to 0$. Ce sera la même pour tous les types de Bianchi pour lesquels nous réutiliserons les mêmes variables $x_\pm$, y et z. La contrainte Hamiltonienne et les équations de champs se réécrivent comme:

$$x_+^2 + x_-^2 + 24y^2 + 12z^2 + 12w^2 = 1 \tag{2.11}$$
$$\dot{x}_+ = 72y^2 x_+ + 24w^2 x_+ - 24w^2 \tag{2.12}$$
$$\dot{x}_- = 72y^2 x_- + 24w^2 x_- - 24\sqrt{3}w^2 \tag{2.13}$$
$$\dot{y} = y(6\ell z + 72y^2 - 3 + 24w^2) \tag{2.14}$$
$$\dot{z} = y^2(72z - 12\ell) + 24w^2 z \tag{2.15}$$
$$\dot{w} = 2w(x_+ + \sqrt{3}x_- + 12w^2 + 36y^2 - 1) \tag{2.16}$$

avec $\ell = \phi U_\phi U^{-1}(3+2\omega)^{-1/2}$. La contrainte montre que les variables (2.7-2.10) sont normalisées. De plus, nous retrouvons l'équation habituelle pour le champ scalaire:

$$\dot{\phi} = \frac{12\phi}{\sqrt{3+2\omega}} z,$$

Afin de déterminer le comportement asymptotique des fonctions nous aurons besoin de connaître le comportement asymptotique de l'Hamiltonien. Nous réécrivons donc l'équation de Hamilton pour H sous la forme:

$$\dot{H} = -H(72y^2 + 24w^2) \tag{2.17}$$

Elle montre que H est une fonction monotone gardant son signe initial au cours de son évolution. Par conséquent, on déduit de la fonction lapse que lorsque H est initialement positif (négatif), $\Omega \to -\infty$ correspond aux époques tardives(respectivement primordiales) et vice-versa lorsque $\Omega \to +\infty$.

Munie de toutes ces équations, nous pouvons désormais calculer les points d'équilibre correspondant à une isotropisation de classe 1 à partir des équations (2.12-2.16). Il en existe plusieurs mais le seul à retenir [1] est tel que:

$$(x_+,x_-,y,z,w) = (0{,}0, \pm\sqrt{3-\ell^2}(6\sqrt{2})^{-1},\ell/6{,}0)$$

Il est donc semblable à celui trouvé pour le modèle plat de type I de Bianchi. Il sera réel et correspondra à un état d'équilibre si ℓ tend vers une constante telle que $\ell^2 < 3$. Afin de trouver le comportement asymptotique de w, on linéarise (2.16) au voisinage de l'équilibre en négligeant les variables w et $x_\pm$ tendant vers zéro. Il vient:

$$w \to e^{(1-\ell^2)\Omega}$$

Linéarisant de la même manière (2.12), utilisant l'hypothèse de variabilité de ℓ^2 et introduisant cette dernière expression pour w, on obtient que $x_\pm$ se comportent comme la somme de deux termes $e^{2(1-\ell^2)\Omega}$ et $e^{(3-\ell^2)\Omega}$. L'isotropie ayant besoin de $x_\pm \to 0$ et $\ell^2 < 3$, nous en déduisons que cela arrive seulement lorsque $\ell^2 < 1$ en $\Omega \to -\infty$. La valeur spéciale $\ell^2 = 1$ n'est pas compatible avec l'isotropie. Ceci découle de notre hypothèse de variabilité de ℓ^2 qui implique que si $\ell^2 \to 1$, $\ell^2 - 1$ disparaît généralement plus vite que Ω^{-1}. Mais alors, w tendrait vers une constante non nulle ce qui est incompatible avec l'expression des points d'équilibre.
Les deux limites $\ell^2 < 1$ et $\Omega \to -\infty$ permettent à $x_\pm$ mais aussi à w de tendre vers zéro. Il vient qu'asymptotiquement

$$x_\pm \to e^{2(1-\ell^2)\Omega}$$

Afin de savoir si notre modèle s'isotropise, il nous faut vérifier que $p_\pm e^{3\Omega} \to 0$ lorsque $\Omega \to -\infty$. Pour cela, on écrit $\dot{p}_\pm/H$ comme une fonction de $x_\pm$ et w et on utilise leurs comportements asymptotiques. On calcul alors que $\dot{p}_\pm/p_\pm$ tend vers la constante $-(1+\ell^2)$. Par conséquent, $p_\pm e^{3\Omega} \to e^{(2-\ell^2)\Omega}$ et disparaît lorsque Ω diverge négativement et que les conditions nécessaires à l'isotropie sont respectées. Les comportements asymptotiques des fonctions métriques et du potentiel sont les mêmes que pour le modèle de Bianchi de type I et dépendent de la même manière de la disparition ou non de la fonction ℓ^2 à l'approche de l'isotropie. La 3-courbure quant à elle tend vers zéro lorsque $\Omega \to -\infty$, montrant que l'Univers devient plat.

2.2.2 Modèles de Bianchi de types VI_0 et VII_0

Cette fois les variables que nous allons utiliser sont:

$$x_\pm = p_\pm H^{-1} \tag{2.18}$$
$$y = \pi R_0^3 e^{-3\Omega} U^{1/2} H^{-1} \tag{2.19}$$
$$z = p_\phi \phi (3+2\Omega)^{-1/2} H^{-1} \tag{2.20}$$
$$w_\pm = \pi R_0^2 e^{-2\Omega + 2\beta_+ \pm 2\sqrt{3}\beta_-} H^{-1} \tag{2.21}$$

La différence avec le modèle de Bianchi de type II est qu'il nous faut 2 variables pour décrire la courbure, dont l'une d'elle (w_+) est la variable w précédemment définie pour ce dernier modèle. Ceci est en accord avec [25] où deux variables N_i sont également nécessaires. La contrainte Hamiltonienne s'écrit:

$$x_+^2 + x_-^2 + 24y^2 + 12z^2 + 12(w_+ \pm w_-)^2 = 1 \tag{2.22}$$

et les équations de champs deviennent

$$\dot{x}_+ = 72y^2 x_+ + 24(x_+ - 1)(w_- \pm w_+)^2 \tag{2.23}$$
$$\dot{x}_- = 72y^2 x_- + 24x_-(w_- \pm w_+)^2 + 24\sqrt{3}(w_-^2 - w_+^2) \tag{2.24}$$

1. Pour une justification de cette sélection le lecteur peut se référer à l'article [127] reproduit en annexe.

$$\dot{y} = y(6\ell z + 72y^2 - 3 + 24(w_- \pm w_+)^2) \quad (2.25)$$
$$\dot{z} = y^2(72z - 12\ell) + 24z(w_- \pm w_+)^2 \quad (2.26)$$
$$\dot{w}_+ = 2w_+ \left[x_+ + \sqrt{3}x_- + 12(w_- \pm w_+)^2 + 36y^2 - 1\right] \quad (2.27)$$
$$\dot{w}_- = 2w_- \left[x_+ - \sqrt{3}x_- + 12(w_- \pm w_+)^2 + 36y^2 - 1\right] \quad (2.28)$$

A nouveau nous exprimons l'équation de Hamilton pour H en fonctions des variables y et $w_\pm$:

$$\dot{H} = -H\left[72y^2 + 24(w_+ \pm w_-)^2\right] \quad (2.29)$$

Dans les équations, les symboles $\pm$ correspondent respectivement aux modèles de Bianchi de type VI_0 et VII_0. Pour le premier modèle, la contrainte (2.22) montre que les variables sont normalisées. Ce n'est pas le cas pour le second: à cause du signe -, w_+ et w_- pourraient diverger si la différence $w_+ - w_-$ reste finie, respectant ainsi la contrainte. Nous montrerons plus bas que ceci est en fait impossible. Supposant que toutes les variables sont normalisées, nous en déduisons que l'isotropisation est impossible pour une valeur finie de Ω. En effet, si $\Omega \to const$ lorsque le temps propre t diverge, $d\Omega/dt \to 0$. Mais de la forme de la fonction lapse et du fait que $dt = -Nd\Omega$, on en déduit que H devrait tendre vers zéro. Il vient alors de la définition des variables $w_\pm$ qu'elles devraient diverger ce qui est incompatible avec le fait qu'elles sont bornées au voisinage de l'isotropie. Ainsi, l'isotropisation ne peut mener l'Univers vers un état statique et Ω diverge forcément.
Une fois de plus, on retrouve les mêmes points d'équilibre que pour le modèle de Bianchi de type II:

$$(x_+,x_-,y,z,w_\pm) = (0,0,\pm\sqrt{3-\ell^2}(6\sqrt{2})^{-1},\ell/6,0)$$

La démonstration est donnée dans [127], reproduit en annexe. Il seront rééls si ℓ^2 tend vers une constante plus petite que 3.
De la même manière que pour le modèle de Bianchi de type II, on peut montrer que Ω est une fonction monotone du temps propre dont la divergence en $-\infty$ correspond aux époques tardives si l'Hamiltonien est initialement positif. On montre également que les comportements asymptotiques des fonctions $x_\pm$, $w_\pm$, $p_\pm e^{3\Omega}$, $e^{-\Omega}$ et U sont les mêmes, imposant que $\ell^2 < 1$ à l'approche de l'isotropie.
L'ensemble de ces résultats a été démontré non pas en considérant les comportements individuels de w_+ et w_- mais en considérant que $w_+ \pm w_- \to 0$. Comme nous déduisons de cette unique limite qu'à l'approche de l'isotropie, $w_\pm \to 0$, il s'ensuit que ces variables sont toujours bornées comme énoncé au début de cette section, et en particulier pour le modèle de Bianchi de type VII_0.

2.2.3 Modèles de Bianchi de types $VIII$ et IX

Nous utiliserons les variables suivantes:

$$x_\pm = p_\pm H^{-1}$$
$$y = \pi R_0^3 e^{-3\Omega} U^{1/2} H^{-1}$$
$$z = p_\phi \phi (3+2\Omega)^{-1/2} H^{-1}$$
$$w_p = \pi R_0^2 e^{-2\Omega+2\beta_+} H^{-1}$$
$$w_m = \pi R_0^2 e^{-2\Omega-2\beta_+} H^{-1}$$
$$w_- = e^{2\sqrt{3}\beta_-}$$

Comme on peut le voir, les variables w_p et w_m ne sont pas indépendantes l'une de l'autre et à l'approche de l'isotropie nous avons $w_p \propto w_m \propto e^{-2\Omega}H^{-1}$. Notons de plus que w_- est une variable positive. Trois variables w sont donc nécessaires pour décrire la courbure de la même manière que trois variables N_i sont utilisées dans [25]. L'équation de contrainte s'écrit alors:

$$x_+^2 + x_-^2 + 24y^2 + 12z^2 + 12[w_p^3(1+w_-^4) \pm 2w_-(w_m w_p)^{3/2}(1+w_-^2) +$$
$$w_-^2(w_m^3 - 2w_p^3)](w_-^2 w_p)^{-1} = 1$$

et les équations de champs:

$$\dot{x}_+ = 72y^2 x_+ + 24\{w_p^3(x_+ - 1)(1+w_-^4) \pm w_-(1+2x_+)(w_m w_p)^{3/2}(1+w_-^2)$$

$$+w_-^2\left[(2+x_+)w_m^3-2(x_+-1)w_p^3\right]\}(w_-^2w_p)^{-1} \quad (2.30)$$

$$\dot{x}_-=72y^2x_-+24\{w_p^3\left[w_-^4(x_--\sqrt{3})+x_-+\sqrt{3})\right]\pm w_-(w_mw_p)^{3/2}[w_-^2$$
$$(-\sqrt{3}+2x_-)+(\sqrt{3}+2x_-)]+w_-^2x_-(w_m^3-2w_p^3)\}(w_-^2w_p)^{-1} \quad (2.31)$$

$$\dot{y}=y\{6\ell z+72y^2-3+24[w_p^3(1+w_-^4)\pm 2(w_mw_p)^{3/2}w_-(1+w_-^2)+$$
$$w_-^2(w_m^3-2w_p^3)](w_-^2w_p)^{-1}\} \quad (2.32)$$

$$\dot{z}=y^2(72z-12\ell)+24z[w_p^3(1+w_-^4)\pm 2(w_mw_p)^{3/2}w_-(1+w_-^2)+$$
$$w_-^2(w_m^3-2w_p^3)](w_-^2w_p)^{-1} \quad (2.33)$$

$$\dot{w}_p=w_p\{-2+2x_++72y^2+24[w_p^3(1+w_-^4)\pm 2w_-(w_mw_p)^{3/2}(1+w_-^2)$$
$$+w_-^2(w_m^3-2w_p^3)](w_-^2w_p)^{-1}\} \quad (2.34)$$

$$\dot{w}_m=w_m\{-2-2x_++72y^2+24[w_p^3(1+w_-^4)\pm 2w_-(w_mw_p)^{3/2}(1+w_-^2)$$
$$+w_-^2(w_m^3-2w_p^3)](w_-^2w_p)^{-1}\} \quad (2.35)$$

$$\dot{w}_-=2\sqrt{3}w_-x_- \quad (2.36)$$

L'équation de Hamilton pour H devient:

$$\dot{H}=-H[72y^2+24(\pm 2\frac{w_p^{1/2}w_m^{3/2}}{w_-}\pm 2w_p^{1/2}w_m^{3/2}w_--2w_p^2+\frac{w_p^2}{w_-^2}+$$
$$w_p^2w_-^2+\frac{w_m^3}{w_p})+\frac{3}{2}(\gamma-2)k^2] \quad (2.37)$$

Le signe $\pm$ représente respectivement le modèle de Bianchi de type $VIII$ ou IX. La contrainte montre que les variables ne sont pas forcément normalisées: si l'une d'elle diverge, cette divergence peut être contrebalancée par celle de w_m ou w_p. Donc si nous montrons que l'isotropie ne se produit que pour des valeurs finies de w_m et w_p, cela signifiera qu'elle ne se produit que pour des valeurs finies de toutes les variables.
Afin d'atteindre ce but, nous écrirons qu'à l'approche de l'isotropie $w_p\to w_m\to w$ et $w_-\to 1$. Alors la contrainte du modèle de Bianchi de type $VIII$ montre que toutes les variables sont positives et donc doivent prendre des valeurs finies. En ce qui concerne le modèle de Bianchi de type IX, supposons que w diverge. Alors si l'on pose $x_\pm=0$, on déduit de la contrainte que $3w^2\to 2y^2+z^2-1/12$ et de l'équation pour $\dot{w}$ que $3w^2\to 3y^2-1/12$, impliquant qu'asymptotiquement $z^2\to y^2$ et divergent comme w^2. Cependant, avec ces limites on obtient des équations pour $\dot{y}$ et $\dot{z}$ que $\dot{y}\to 6\ell z^2-3z$ et $\dot{z}\to -12\ell z^2+2z$. Alors l'équilibre pour y et z peut seulement être obtenu lorsque $z\to 0$ ce qui est en contradiction avec la divergence de z que nous venons de montrer. On en déduit donc qu'un état d'équilibre isotrope stable est impossible si w_p et w_m divergent. Il s'ensuit pour les mêmes raisons que pour les modèles de Bianchi précédents, que l'isotropisation est impossible pour une valeur finie de Ω.
On peut aussi montrer que w_p et w_m ne peuvent pas tendre vers des constantes non nulles. Supposons que ce soit le cas et définissons les deux constantes w et α telles que $w_p\to w$ et $w_m\to\alpha w$. On introduit ces limites dans les équations pour $\dot{x}_\pm$ avec $x_\pm=0$. Il vient:

$$\dot{x}_+=-24w^2(1+w_-\alpha^{3/2}(1+w_-^2)-2w_-^2(1+\alpha^3)+w_-^4)w_-^{-2} \quad (2.38)$$

$$\dot{x}_-=-24\sqrt{3}w^2(w_-^2-1)(1-\alpha^{3/2}w_-+w_-^2)w_-^{-2} \quad (2.39)$$

Alors, pour le modèle de Bianchi de type $VIII$, on en déduit que l'équilibre pour $x_\pm$ sera atteint uniquement si α tend vers la valeur complexe $(-1)^{2/3}$ ou/et si w_- est négatif ce qui est impossible. Pour le modèle de Bianchi de type IX, l'équilibre pour $x_\pm$ peut être atteint si $w_p\to w_m$ (i.e. $\beta_\pm\to 0$) et $w_-\to 1$. Alors, calculant les points d'équilibre, les seuls qui soient réels et tels que w_p et w_m soient différents de 0 sont $(x_+,x_-,y,z,w_p,w_m,w_-)=(0,0,\pm(6\ell)^{-1},(6\ell)^{-1},\pm(1-\ell^2)^{1/2}(6\ell)^{-1},1)$. Ils vérifient l'équation de contrainte et sont réels si $\ell^2<1$. De plus, on calcule que w_p et w_m tendent vers $\pm(1-\ell^2)^{1/2}(1-e^{\frac{4\Omega(\ell^2-1)+\omega_0}{\ell^2}}+36\ell^2)^{-1/2}$ et donc atteignent l'équilibre en $\Omega\to+\infty$. Introduisant ces expressions dans $\dot{x}_+$, il vient alors que x_+ tend vers une valeur complexe en $\Omega\to+\infty$ et donc que ces points d'équilibre sont exclus.
Par conséquent, les seuls points d'équilibre isotropes possibles sont tels que

$$(x_+,x_-,y,z,w_p,w_m,w_-)=(0,0,\pm\sqrt{3-\ell^2}(6\sqrt{2})^{-1},\ell/6,0,0,1)$$

Les variables w_m et w_p se comportent asymptotiquement comme $e^{(1-\ell^2)\Omega}$ et $x_\pm$ comme $e^{2(1-\ell^2)\Omega}$. Il s'ensuit que les comportements asymptotiques des fonctions métriques et du potentiel sont les mêmes asymptotiquement que pour les autres modèles. En revanche, le signe de l'Hamiltonien (2.37) n'est pas conservé tout au long de l'évolution temporelle et il n'est donc pas possible de savoir si la limite $\Omega \to -\infty$ correspond aux époques tardives ou primordiales.

2.2.4 Discussion

Techniquement par rapport au modèle de Bianchi de type I, il existe plusieurs différences:

- Mise à part pour les modèles de Bianchi de type II et VI_0, la contrainte n'implique pas automatiquement que les variables x, y, z et w soient bornées. Il faut montrer que c'est le cas à l'approche d'un état isotrope stable.
- Il faut montrer que l'isotropie correspond à une expansion infinie de l'Univers ($\Omega \to -\infty$).
- Il faut montrer que le produit $p_\pm e^{3\Omega}$ tend vers zéro.

Physiquement, les modèles avec courbure sont plus intéressants que les modèles à sections spatiales plates car ils permettent de montrer que l'isotropisation de classe 1 s'accompagne d'une expansion accélérée et d'un aplatissement des sections spatiales. Ceci provient du fait que les points d'équilibre sont tels que les variables w liées à la courbure disparaissent à l'approche de l'isotropie, réduisant l'intervalle de valeurs dans lequel la fonction ℓ doit tendre asymptotiquement afin de permettre l'isotropisation. Les comportements asymptotiques des fonctions métriques et du potentiel sont alors les mêmes que pour le modèle de Bianchi de type I car l'Hamiltonien et la fonction lapse des modèles avec courbure se comportent asymptotiquement de la même manière. On a donc le résultat suivant:

Soit une théorie tenseur-scalaire minimalement couplée et massive et la quantité ℓ définie par $\ell = \frac{\phi U_\phi}{U(3+2\omega)^{1/2}}$. Le comportement asymptotique du champ scalaire à l'approche de l'isotropie est donné par la forme asymptotique de la solution de l'équation différentielle $\dot{\phi} = 2\frac{\phi^2 U_\phi}{U(3+2\omega)}$ en $\Omega \to -\infty$. Cette limite ne correspond pas forcément aux époques tardives pour les modèles de Bianchi de type $VIII$ et IX contrairement aux autres modèles. Une condition nécessaire à l'isotropisation de classe 1 est que $\ell^2 < 1$. Si ℓ tend vers une constante non nulle, les fonctions métriques tendent vers $t^{\ell^{-2}}$ et le potentiel disparaît comme t^{-2}. Si ℓ tend vers zéro, l'Univers tend vers un modèle de De Sitter et le potentiel vers une constante. Dans tous les cas l'Univers est asymptotiquement en expansion accélérée et s'aplatit.

Ainsi, le comportement accéléré de l'Univers et son aplatissement pourraient trouver une explication naturelle à travers le fait que l'Univers s'isotropise. Remarquons que le comportement asymptotique du modèle de Bianchi de type IX n'est pas oscillatoire. Ceci n'est pas incompatible avec un comportement de type mixmaster au voisinage d'une singularité comme observé dans [128]. Notons également que le fait qu'il n'existe qu'un seul état équilibre isotrope tel que la courbure tende vers zéro peut paraître choquant. Ceci pourrait être dû au fait que nous appliquons l'hypothèse de variabilité de ℓ.

2.3 Avec fluide parfait

En l'absence de courbure, nous avons vu qu'en présence d'un fluide parfait dépourvu de couplage avec le champ scalaire, lorsque $k \to const \neq 0$, une expansion accélérée de l'Univers était impossible car les fonctions métriques tendent vers $t^{\frac{2}{3\gamma}}$. Au contraire, dans la section précédente, nous avons vu qu'en présence de courbure, l'expansion de l'Univers aux époques tardives était toujours accélérée lors de l'isotropisation. Le but de cette section est donc de savoir ce qui se passe lorsque l'on considère à la fois de la courbure et un fluide parfait d'équation d'état $p = (\gamma - 1)\rho$.

En ce qui concernent les équations de champs, elles changent peu: un terme contenant la variable k que nous avions précédement définie par $k^2 = \delta e^{3(\gamma-2)\Omega} H^{-2}$ (cf équation (1.20)), vient s'ajouter dans chaque équation de champs. Nous les avons réécrites dans l'annexe à la fin de cette section. Ci-dessous, on examine le processus d'isotropisation lorsque le paramètre de densité du fluide parfait tend vers zéro ($k \to 0$) ou vers une constante non nulle ($k \not\to 0$).

$\underline{k \to 0}$

Les résultats sont les mêmes qu'en l'absence d'un fluide parfait mais la condition $k \to 0$, indiquant que $U >> V^{-\gamma}$, ajoute une nouvelle contrainte. On peut cependant montrer que celle ci est moins restrictive que la contrainte $\ell^2 < 1$ nécessaire à l'isotropisation. Par conséquent, contrairement à ce qui se passait pour le modèle de Bianchi de type I, elle ne modifie pas cet intervalle de valeurs pour ℓ^2.

$k \not\to 0$

La première question à se poser est de savoir si la condition $p_\pm e^{3\Omega} \to 0$, nécessaire à l'isotropie peut être respectée lorsque Ω tend vers une constante. Supposons que ce soit le cas, alors il faudrait que $p_\pm \to 0$. Supposons dans le même temps que $x_\pm \not\to 0$. Alors d'après la définition de $x_\pm$, il faudrait que l'Hamiltonien H soit tel que $H \to 0$. Mais alors k divergerait et la contrainte ne serait pas respectée car, à l'approche de l'isotropie, toutes les variables doivent être bornées comme montré dans l'article [129] reproduit en annexe. Donc, H ne peut pas tendre vers zéro et $x_\pm$ doit disparaître à l'approche de l'isotropie. De la même manière H ne peut diverger car alors $k \to 0$ ce qui est en désaccord avec notre supposition de départ. Par conséquent, lorsque l'isotropisation se produit pour une valeur finie de Ω, H doit tendre vers une quantité finie et non nulle et, d'après leurs définitions, il doit donc en être de même pour les variables w décrivant la courbure.

Or, lorsque l'on calcule les points d'équilibre des équations de champs tels que $x_\pm \to 0$, les seuls points possible sont:

$$(x_\pm,y,z,w) = (0, \pm \frac{\sqrt{\gamma(2-\gamma)}}{4\sqrt{2}\pi R_0^3 \ell}, \frac{\gamma}{4\ell}, 0)$$

et sont donc tels que $w \to 0$. Il s'ensuit que l'isotropisation ne peut se produire que pour une valeur infinie de Ω. La contrainte hamiltonienne montre également que $k^2 = 1 - \frac{3\gamma}{2\ell^2}$, cette variable étant finie et réelle si $\ell^2 > \frac{3}{2}\gamma$. On calcule alors qu'asymptotiquement:

$$H \to e^{-\frac{3}{2}(2-\gamma)\Omega}$$

ce qui montre bien que $k^2 \to const \neq 0$ et, d'après la définition (1.21) de k, que $U \propto V^{-\gamma}$. Les variables w quant à elles se comportent asymptotiquement comme:

$$w \to e^{(1-\frac{3\gamma}{2})\Omega}$$

Or $\gamma \in [1,2]$ et donc w tendra vers zéro si $\Omega \to +\infty$. Cependant, pour les variables $x_\pm$ on calcule que

$$x \to e^{(2-3\gamma)\Omega}(e^{(1+\frac{3\gamma}{2})\Omega} + x_0)$$

Nous voyons alors que pour cet intervalle de γ et cette limite pour Ω, $x_\pm$ divergent. Donc, le point d'équilibre ne peut être atteint dans ces conditions. Parallèlement, connaissant $x_\pm$ et H, on calcule que

$$p_\pm \to e^{-\frac{1}{2}(2+3\gamma)\Omega} + cte$$

Par conséquent, $p_\pm e^{3\Omega}$, w et $x_\pm$ disparaîtront si $\gamma < 2/3$ et $\Omega \to -\infty$. Alors, à l'approche de l'isotropie, $e^{-\Omega} \to t^{\frac{2}{3\gamma}}$ et $U \to t^{-2}$ comme pour le modèle de Bianchi de type I. Cette restriction sur γ n'existait pas pour le modèle de Bianchi de type I et ne correspond pas à un fluide parfait ordinaire.

2.3.1 Application

Pour résumer, nous avons le résultat suivant:

Lorsque $\Omega_m \to 0$, l'isotropisation se produit de la même manière qu'en l'absence de fluide parfait. En revanche elle est impossible lorsque $\Omega_m \not\to 0$ pour un fluide parfait ordinaire sauf si $\gamma < 2/3$.

Pour finir, nous reprenons l'application de la section 1.1.4 avec le potentiel en exponentiel du champ scalaire $e^{m\phi}$. Désormais la limite sur m permettant l'isotropisation est $m^2 < 2$ ce que confirme une simulation numérique montrant sur la figure 2.1 l'évolution des variables lorsque $m^2 < 2$ et $m^2 > 2$ pour le modèle de Bianchi de type II. Dans le premier cas, $x_\pm \to 0$ et l'Univers s'isotropise alors que dans le second ces variables tendent vers une constante démontrant une croissante linéaire des fonctions $\beta_\pm$ par rapport à Ω.

2.4 Annexe: équations de champs des modèles de Bianchi avec courbure et fluide parfait

Bianchi type II

La contrainte Hamiltonienne s'écrit:

$$x_+^2 + x_-^2 + 24y^2 + 12z^2 + 12w^2 + k^2 = 1 \tag{2.40}$$

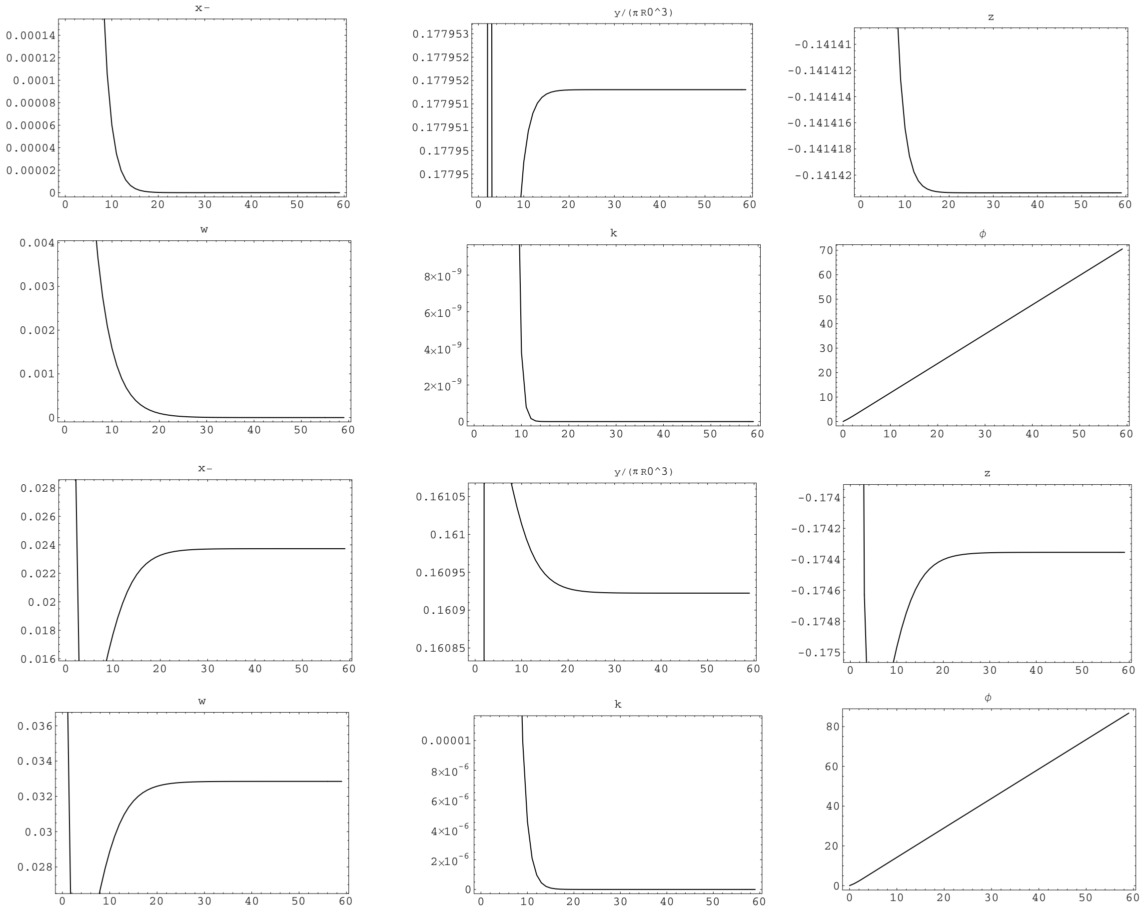

FIG. 2.1 – *Evolution des variables x, y et z lorsque $\frac{(3+2\omega)^{1/2}}{\phi} = \sqrt{2}$, $U = e^{m\phi}$, $R_0^3 = (\sqrt{24}\pi)^{-1}$ avec les valeurs initiales $(x_+,x_-,y,z,w,\phi) = (0.87,0.06,0.025,-0.12,0.065,0.014)$. Les 6 premières figures sont telles que $m = -1.2$(l'évolution de x_+ est semblable à celle de x_-): l'Univers s'isotropise. La deuxième série de 6 figures est telle que $m = -1.5$: l'Univers ne s'isotropise plus.*

Les équations de Hamilton sont:

$$\dot{x}_+ = 72y^2 x_+ + 24w^2 x_+ - 24w^2 - 3/2(\gamma - 2)k^2 x_+ \tag{2.41}$$

$$\dot{x}_- = 72y^2 x_- + 24w^2 x_- - 24\sqrt{3}w^2 - 3/2(\gamma - 2)k^2 x_- \tag{2.42}$$

$$\dot{y} = y(6\ell z + 72y^2 - 3 + 24w^2) - 3/2(\gamma - 2)k^2 y \tag{2.43}$$

$$\dot{z} = y^2(72z - 12\ell) + 24w^2 z - 3/2(\gamma - 2)k^2 z \tag{2.44}$$

$$\dot{w} = 2w(x_+ + \sqrt{3}x_- + 12w^2 + 36y^2 - 1) - 3/2(\gamma - 2)k^2 w \tag{2.45}$$

et l'équation pour le champ scalaire, commune à tous les modèles de Bianchi, s'écrit:

$$\dot{\phi} = 12\frac{z\phi}{\sqrt{3 + 2\omega}} \tag{2.46}$$

L'équation pour $\dot{H}$ peut être réécrite comme:

$$\dot{H} = -H(72y^2 + 24w^2 + \frac{3}{2}(\gamma - 2)k^2) \tag{2.47}$$

Bianchi VI_0 et VII_0 models
La contrainte Hamiltonienne s'écrit:

$$x_+^2 + x_-^2 + 24y^2 + 12z^2 + 12(w_+ \pm w_-)^2 + k^2 = 1 \tag{2.48}$$

On a donc les équations de Hamilton suivantes:

$$\dot{x}_+ = 72y^2x_+ + 24(x_+ - 1)(w_- \pm w_+)^2 - 3/2(\gamma - 2)k^2x_+ \tag{2.49}$$
$$\dot{x}_- = 72y^2x_- + 24x_-(w_- \pm w_+)^2 + 24\sqrt{3}(w_-^2 - w_+^2) - 3/2(\gamma - 2)k^2x_- \tag{2.50}$$
$$\dot{y} = y(6\ell z + 72y^2 - 3 + 24(w_- \pm w_+)^2) - 3/2(\gamma - 2)k^2y \tag{2.51}$$
$$\dot{z} = y^2(72z - 12\ell) + 24z(w_- \pm w_+)^2 - 3/2(\gamma - 2)k^2z \tag{2.52}$$
$$\dot{w}_+ = 2w_+\left[x_+ + \sqrt{3}x_- + 12(w_- \pm w_+)^2 + 36y^2 - 1\right] - 3/2(\gamma - 2)k^2w_+ \tag{2.53}$$
$$\dot{w}_- = 2w_-\left[x_+ - \sqrt{3}x_- + 12(w_- \pm w_+)^2 + 36y^2 - 1\right] - 3/2(\gamma - 2)k^2w_- \tag{2.54}$$

L'équation pour $\dot{H}$ est:

$$\dot{H} = -H\left[72y^2 + 24(w_+ \pm w_-)^2 + \frac{3}{2}(\gamma - 2)k^2\right] \tag{2.55}$$

<u>Bianchi $VIII$ et IX models</u>
La contrainte Hamiltonienne s'écrit:

$$x_+^2 + x_-^2 + 24y^2 + 12z^2 + 12[w_p^3(1 + w_-^4) \pm 2w_-(w_mw_p)^{3/2}(1 + w_-^2) + \\ w_-^2(w_m^3 - 2w_p^3)](w_-^2w_p)^{-1} + k^2 = 1$$

Les équations de Hamilton sont:

$$\dot{x}_+ = 72y^2x_+ + 24\{w_p^3(x_+ - 1)(1 + w_-^4) \pm w_-(1 + 2x_+)(w_mw_p)^{3/2}(1 + w_-^2) \\ +w_-^2\left[(2 + x_+)w_m^3 - 2(x_+ - 1)w_p^3\right]\}(w_-^2w_p)^{-1} - 3/2(\gamma - 2)k^2x_+ \tag{2.56}$$
$$\dot{x}_- = 72y^2x_- + 24\{w_p^3\left[w_-^4(x_- - \sqrt{3}) + x_- + \sqrt{3})\right] \pm w_-(w_mw_p)^{3/2}[w_-^2 \\ (-\sqrt{3} + 2x_-) + (\sqrt{3} + 2x_-)] + w_-^2x_-(w_m^3 - 2w_p^3)\}(w_-^2w_p)^{-1} - 3/2(\gamma - 2)k^2x_- \tag{2.57}$$
$$\dot{y} = y\{6\ell z + 72y^2 - 3 + 24[w_p^3(1 + w_-^4) \pm 2(w_mw_p)^{3/2}w_-(1 + w_-^2) + \\ w_-^2(w_m^3 - 2w_p^3)](w_-^2w_p)^{-1}\} - 3/2(\gamma - 2)k^2y \tag{2.58}$$
$$\dot{z} = y^2(72z - 12\ell) + 24z[w_p^3(1 + w_-^4) \pm 2(w_mw_p)^{3/2}w_-(1 + w_-^2) + \\ w_-^2(w_m^3 - 2w_p^3)](w_-^2w_p)^{-1} - 3/2(\gamma - 2)k^2z \tag{2.59}$$
$$\dot{w}_p = w_p\{-2 + 2x_+ + 72y^2 + 24[w_p^3(1 + w_-^4) \pm 2w_-(w_mw_p)^{3/2}(1 + w_-^2) \\ +w_-^2(w_m^3 - 2w_p^3)](w_-^2w_p)^{-1}\} - 3/2(\gamma - 2)k^2w_p \tag{2.60}$$
$$\dot{w}_m = w_m\{-2 - 2x_+ + 72y^2 + 24[w_p^3(1 + w_-^4) \pm 2w_-(w_mw_p)^{3/2}(1 + w_-^2) \\ +w_-^2(w_m^3 - 2w_p^3)](w_-^2w_p)^{-1}\} - 3/2(\gamma - 2)k^2w_m \tag{2.61}$$
$$\dot{w}_- = 2\sqrt{3}w_-x_- \tag{2.62}$$

et pour $\dot{H}$

$$\dot{H} = -H[72y^2 + 24(\pm 2\frac{w_p^{1/2}w_m^{3/2}}{w_-} \pm 2w_p^{1/2}w_m^{3/2}w_- - 2w_p^2 + \frac{w_p^2}{w_-^2} + \\ w_p^2w_-^2 + \frac{w_m^3}{w_p}) + \frac{3}{2}(\gamma - 2)k^2] \tag{2.63}$$

Chapitre 3

Isotropisation et quintessence

3.1 Introduction

Dans les sections précédentes nous avons étudié l'isotropisation des modèles cosmologiques homogènes de Bianchi en théories tenseur-scalaires. Ici nous utilisons ces résultats pour montrer la relation entre l'isotropisation de ces modèles et le phénomène de quintessence, indiquant ainsi que la quintessence peut être l'aboutissement naturel d'un processus d'isotropisation de classe 1.
La quintessence est l'un des moyens d'expliquer pourquoi l'expansion de l'Univers subit actuellement une accélération depuis un redshift estimé entre 0.5 et 1[130]. En effet, lorsque le champ scalaire est quintessent ou le devient (trackers theories), il peut être équivalent à la présence d'un fluide parfait avec une pression négative, pouvant ainsi provoquer l'accélération en question. Les champs scalaires ne sont pas les seuls à pouvoir l'expliquer. Des théories branaires[131] ou encore celles faisant intervenir des termes de courbure d'ordre supérieurs à celui du scalaire de Ricci peuvent aussi la provoquer. Il faut cependant noter que ce dernier type de théorie peut se ramener à une théorie tenseur-scalaire, moyennant une transformation conforme[112, 26]. Cette liste n'est bien sûr pas exhaustive. Toutefois l'attrait des champs scalaires est grand de par leur l'omniprésence en physique des particules à travers, par exemple, le mécanisme de Higgs ou encore la supersymétrie.
Jusqu'à précédement, nous avons principalement étudié les aspects mathématiques et dynamiques de l'isotropisation en cherchant à déterminer les états d'équilibre isotropes stables et la forme asymptotique des fonctions métriques. Dans ce chapitre, nous souhaitons examiner l'aspect physique de ces résultats en définissant sous quelles conditions ces champs peuvent devenir quintessents lors de l'isotropisation, en déterminant le redshift pour lequel la densité d'énergie du champ scalaire domine celle de la matière et en déterminant l'évolution asymptotique de l'anisotropie en fonction du redshift z.
Le plan de cette section est le suivant. Dans la section 3.2, nous étudions ce qui se passe en présence d'un champ scalaire minimalement couplé. Dans la section 3.3, nous considérons deux champs scalaires minimalement couplés et montrons que cette théorie semble physiquement indiscernable de celle avec un seul champ scalaire à l'approche de l'isotropie. Dans la section 3.4, nous considérons un champ scalaire non minimalement couplé. Les choses sont alors considérablement plus compliquées car postuler l'isotropisation ne permet pas de prévoir l'évolution asymptotique de la fonction de gravitation G, représentant le couplage non minimal entre le champ scalaire et la courbure. Des arguments en faveur d'un champ scalaire asymptotiquement quintessent sont présentés mais on ne peut aller au delà sans préciser G.

3.2 Avec un champ scalaire minimalement couplé

Dans cette première section, on considère un champ scalaire minimalement couplé dont on rappelle la forme de l'action:

$$S = (16\pi)^{-1}\int [R-(3/2+\omega)\phi^{,\mu}\phi_{,\mu}\phi^{-2}-U+16\pi c^4 L_m]\sqrt{-g}d^4x$$

3.2.1 Détermination de la densité d'énergie et de la pression d'un champ scalaire

Pour établir la nature quintessente ou non d'un champ scalaire, il nous faut écrire son tenseur d'énergie-impulsion sous une forme identique à celle qu'aurait un fluide parfait afin de calculer son indice barotropique. La démonstration est classique pour le cas d'un champ scalaire mais nous la rappelons ici à des fins de comparaison avec les cas moins classiques qui vont suivre. Soit les équations de champs:

$$G_{\alpha\beta} = T_{\alpha\beta(m)} + T_{\alpha\beta(\phi)}$$

où $G_{\alpha\beta}$ est le tenseur d'Einstein et les $T_{\alpha\beta}$ indicés (m) et (ϕ) sont respectivement les tenseurs d'énergie-impulsion du fluide parfait et du champ scalaire. Le premier s'écrit:

$$T_{\alpha\beta(m)} = (\rho_m + p_m)u_\alpha u_\beta + p_m g_{\alpha\beta} \tag{3.1}$$

Le vecteur u est un vecteur de genre temps tel que $g^{\alpha\beta}u_\alpha u_\beta = -1$. Les quantités ρ_m et p_m sont respectivement la densité et la pression d'un fluide parfait dont l'équation d'état est $p_m = w_m \rho_m$, w_m étant une constante appelée indice barotropique. Le tenseur d'énergie-impulsion du champ scalaire est quant à lui défini par:

$$T_{\alpha\beta(\phi)} = \frac{1}{2}\frac{3+2\omega}{\phi^2}\phi_{,\alpha}\phi_{,\beta} - \frac{1}{4}\frac{3+2\omega}{\phi^2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}g_{\alpha\beta} - \frac{1}{2}Ug_{\alpha\beta} \tag{3.2}$$

Afin de trouver la densité ρ_ϕ et la pression p_ϕ relative au champ scalaire tel que son tenseur d'énergie-impulsion prenne la forme de celui d'un fluide parfait, il nous faut donc en premier lieu déterminer le vecteur de genre temps relatif au champ scalaire puis par identification entre $T_{\alpha\beta(m)}$ et $T_{\alpha\beta(\phi)}$, déterminer ρ_ϕ et p_ϕ. On défini le vecteur de genre temps suivant:

$$u_\alpha = \frac{\phi_{,\alpha}}{\sqrt{-g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}}} \tag{3.3}$$

On en déduit que

$$\phi_{,\alpha}\phi_{,\beta} = -g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}u_\alpha u_\beta$$

que l'on introduit dans $T_{\alpha\beta(\phi)}$ pour obtenir:

$$T_{\alpha\beta(\phi)} = -\frac{1}{2}\frac{3+2\omega}{\phi^2}g^{\mu\nu\phi_{,\mu}\phi_{,\nu}}u_\alpha u_\beta - \frac{1}{4}\frac{3+2\omega}{\phi^2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}g_{\alpha\beta} - \frac{1}{2}Ug_{\alpha\beta}$$

Par comparaison avec $T_{\alpha\beta(m)}$, on en déduit le système d'équations nous permettant de calculer p_ϕ et ρ_ϕ:

$$\begin{aligned} p_\phi &= \frac{1}{4}\frac{3+2\omega}{\phi^2}\phi'^2 - \frac{1}{2}U \\ \rho_\phi + p_\phi &= \frac{1}{2}\frac{3+2\omega}{\phi^2}\phi'^2 \end{aligned}$$

d'où il vient:

$$\begin{aligned} 2\rho_\phi &= \frac{1}{2}\frac{3+2\omega}{\phi^2}\phi'^2 + U \\ 2p_\phi &= \frac{1}{2}\frac{3+2\omega}{\phi^2}\phi'^2 - U \end{aligned}$$

A l'aide de ces expressions, nous allons examiner dans quelles conditions le champ scalaire, lorsque l'Univers subit une isotropisation de classe 1, est ou non quintessent en présence ou non de matière et de courbure.

3.2.2 Isotropisation de classe 1 et quintessence

Dans cette section, on va montrer que lorsque l'Univers subit une isotropisation de classe 1, le champ scalaire peut être quintessent. Pour cela, on réécrit ρ_ϕ et p_ϕ à l'aide des variables du formalisme Hamiltonien. Il vient:

$$\rho_\phi = \frac{H^2 e^{6\Omega}}{288\pi^2 R_0^6}\frac{3/2+\omega}{\phi^2}\dot{\phi}^2 + U/2 \tag{3.4}$$

$$p_\phi = \frac{H^2 e^{6\Omega}}{288\pi^2 R_0^6}\frac{3/2+\omega}{\phi^2}\dot{\phi}^2 - U/2 \tag{3.5}$$

On peut alors définir $w(\phi)$ tel que $p_\phi = w_\phi(\phi)\rho_\phi$ et le champ scalaire est quintessent si $w_\phi(\phi)$ tend vers une constante négative. On parle dans ce cas de tracking solution. Ci-dessous, on examine chaque modèle de Bianchi.

Modèle de Bianchi avec sections spatiales plates et sans matière

On rappelle les résultats obtenus dans le chapitre 1 lorsque l'hypothèse de variabilité de ℓ est appliquée:

- ℓ^2 tend vers une constante plus petite que 3.
- Lorsque ℓ tend vers une constante non nulle, les fonctions métriques et le potentiel tendent respectivement vers $t^{\ell^{-2}}$ et t^{-2}. Lorsque ℓ tend vers 0, l'Univers tend vers un modèle de De Sitter avec constante cosmologique.
- $x \to x_0 e^{(3-\ell^2)\Omega}$, $\dot{\phi} \to 2\frac{\phi^2 U_\phi}{(3+2\omega)U}$, $y^2 \to \frac{3-\ell^2}{72\pi^2 R_0^6}$

Par conséquent, on calcule que:

$$H^2 e^{6\Omega} = x_0^{-2} e^{2\ell^2\Omega}$$
$$\frac{3/2+\omega}{\phi^2}\dot{\phi}^2 = 2\ell^2 \tag{3.6}$$

et

$$U = \frac{3-\ell^2}{72\pi^2 R_0^6 x_0^2} e^{2\ell^2\Omega}$$

d'où l'on déduit:

$$\rho_\phi = \frac{3}{144\pi^2 R_0^6 x_0^2} e^{2\ell^2\Omega} \propto t^{-2}$$
$$p_\phi = \frac{2\ell^2 - 3}{144\pi^2 R_0^6 x_0^2} e^{2\ell^2\Omega} \propto t^{-2}$$

Donc, à l'approche de l'isotropie, le champ scalaire se comporte comme un fluide parfait d'équation d'état barotrope $p_\phi = w_\phi \rho_\phi$ avec $w_\phi = \frac{2}{3}\ell^2 - 1 \in [-1,1]$ et la fonction du champ scalaire ℓ peut être asymptotiquement interprétée comme l'indice barotropique de ce fluide. Il sera quintessent si $\ell^2 < 3/2$ ce qui est compatible avec l'isotropisation. Les données de WMAP [1] indiquant que $w_\phi < -0.78$, nous obtenons que $\ell^2 < 0.33$ et que le paramètre de décélération q est tel que $q = \ell^2 - 1 < -0.67$. On peut également calculer $\rho_\phi(z)$. En effet, par définition $\frac{R_0}{R_e} = 1 + z$ et on a donc:

$$\rho_\phi = \rho_{\phi_0}(1+z)^{2\ell^2}$$

ρ_{ϕ_0} étant la densité du champ scalaire pour les époques actuelles. Cette loi est illustrée sur la figure 3.1. Elle est nettement sensible au paramètre ℓ. Plus il est grand, plus la densité d'énergie décroît rapidement. Notons qu'une loi identique correspond à l'évolution du potentiel en fonction du redshift, qui peut ainsi être reconstruit pour les époques tardives. Déterminons la valeur de ρ_{ϕ_0}. On a:

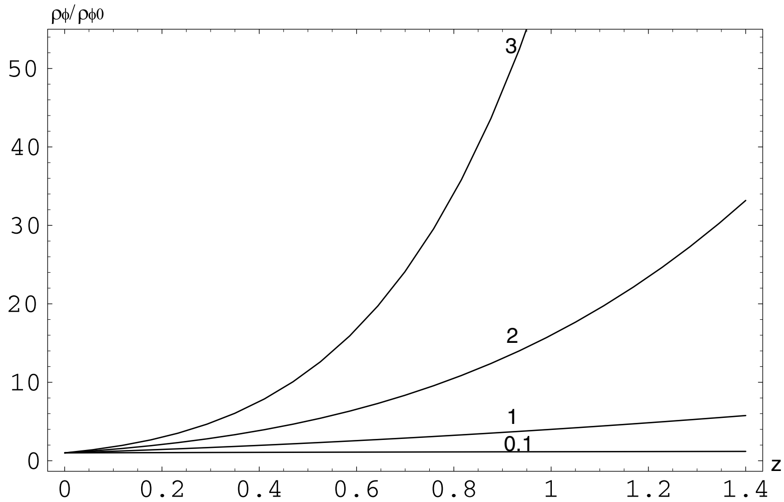

FIG. 3.1 – *Cette figure représentent l'évolution de la densité d'énergie du champ scalaire ρ_ϕ/ρ_{ϕ_0} pour les petits redshift. Chaque courbe est libellée par une valeur de ℓ^2, la courbe supérieure correspondant à la valeur maximale autorisée pour ℓ, soit $\ell^2 = 3$.*

$$\rho_{crit} = \frac{3H_0^2}{8\pi G} = 9.444.10^{-30} g cm^{-3} \tag{3.7}$$

1. http://lambda.gsfc.nasa.gov/

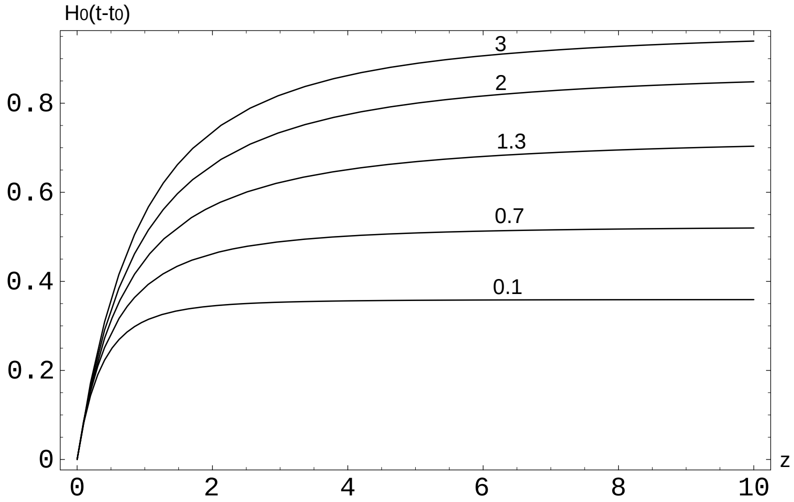

FIG. 3.2 – *Cette figure représente le look back time sans dimension $H_0(t - t_0)$ pour un modèle plat, avec champ scalaire et poussière tel que $\Omega_{m_0} = 0.27$, $\Omega_{\phi_0} = 0.73$ et libellé par ℓ^2.*

avec $H_0 = 71kms^{-1}Mpc^{-1} = 2.297.10^{-18}s^{-1}$ et[2] $G = 6.673.10^{-8}cm^3g^{-1}s^{-2}$. Par conséquent $\rho_{\phi_0} = \Omega_{\phi_0}\rho_{crit} = 0.73\rho_{crit} = 6.894.10^{-30}g/cm^3$.
Pour résumer, les observations de WMAP lorsque l'Univers s'isotropise sont compatibles avec un champ scalaire quintessent dont la densité d'énergie évolue comme $\rho_\phi = \rho_{\phi_0}(1+z)^{2\ell^2}$ avec $\rho_{\phi_0} = 6.894.10^{-30}g/cm^3$ et $\ell^2 < 0.33$.

Modèles de Bianchi avec courbure et sans matière

A l'approche de l'isotropie, les nouveaux résultats à prendre en compte par rapport au modèle sans courbure sont:

- ℓ^2 tend vers une constante plus petite que 1.
- $H \to e^{(\ell^2-3)\Omega}$.

Par conséquent, reprenant les calculs de la section précédente, la densité d'énergie du champ scalaire se comporte de la même manière mais l'indice barotropique se trouve cette fois dans l'intervalle $w_\phi \in [-1, -1/3]$: le champ scalaire est toujours quintessent.

Modèle de Bianchi avec sections spatiales plates et matière tel que $\Omega_m \to 0$

Ce qui change à l'approche de l'isotropie par rapport à la section 3.2.2 ou la matière est absente, est que la condition $\Omega_m \to 0$ impose que $\ell^2 < \frac{3}{2}\gamma$. Les calculs portant sur la densité d'énergie et la pression du champ scalaire sont donc les mêmes qu'en l'absence de fluide parfait mais désormais $w_\phi \in [-1, \gamma - 1]$: en fonction de la valeur de γ le champ scalaire sera (si $\gamma \leq 1$) ou non (si $\gamma \geq 1$) systématiquement quintessent. Calculons le look back time pour un Univers composé de CDM ($\gamma = 1$) et d'un champ scalaire tel qu'aujourd'hui $\Omega_{m_0} = 0.27$ et $\Omega_{\phi_0} = 0.73$. L'Univers étant isotrope, on peut écrire que:

$$H^2 = H_0^2\left[\Omega_{m_0}(1+z)^3 + \Omega_{\phi_0}(1+z)^{2\ell^2}\right] = H_0^2E(z)$$

Le look back time est alors $H_0(t - t_0) = \int_0^z 1/\left[(1+z)E(z)^{1/2}\right]$ et est représenté par la figure 3.2.
Une autre quantité intéressante à représenter est la distance luminosité sans dimension qui s'exprime comme $d_lH_0/c = (1+z)\int_0^z 1/\left[(1+z)E(z)^{1/2}\right]$. La figure 3.3 ci-dessous présente 2 graphes, le premier correspondant à la distance d_lH_0/c et le second à la différence entre cette quantité et la même quantité en l'absence de champ scalaire tel que $\Omega_{m_0} = 1$. On voit à quel point ces distances peuvent être différentes sauf bien sûr aux alentour de $\ell^2 = 3/2$, valeur pour laquelle les densités d'énergie de la matière et du champ scalaire décroissent au même rythme.
Calculons l'époque de la domination du champ scalaire sur la matière en cherchant pour quelle valeur de z on a $\Omega_{m_0}(1+z)^3 = \Omega_{\phi_0}(1+z)^{2\ell^2}$. On obtient la figure 3.4 qui décrit la courbe $\ell(z)^2$ vérifiant cette égalité. Pour $\ell^2 = 0.33$ correspondant à $w_\phi = -0.78$, l'époque de la domination du champ scalaire sur la

2. $1pc = 3.26al = 3.09.10^{13}$kms

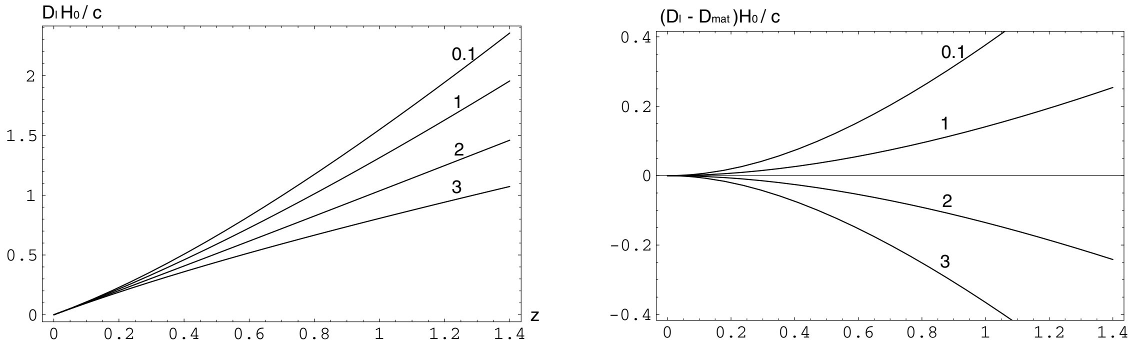

FIG. 3.3 – *La première figure représente la distance luminosité sans dimension pour différentes valeurs de ℓ. La seconde représente la différence entre cette distance et celle obtenue en l'absence de champ scalaire avec $\Omega_{m_0} = 1$.*

matière commence en $z = 0.53$ et à des valeurs plus petites si $\ell^2 < 0.33$. Pour une constante cosmologique obtenue pour $\ell = 0$, ceci se produit pour $z = 0.39$. Lorsque $\ell^2 \to 3\gamma/2$ (ici avec $\gamma = 1$ puisque l'on considère l'équation d'état de la poussière pour représenter la CDM), z tend vers l'infini. Au delà de cette valeur, z est négatif. Pour un Univers isotrope et en accélération correspondant à $\ell^2 < 1$, on doit donc avoir $z < 1.70$. Cette valeur est donc le redshift maximum correspondant à la domination du champ scalaire sur la matière pour un Univers subissant une expansion accélérée et est naturellement petite compte tenu des valeurs observées des densités d'énergies de la matière et du champ scalaire. Nous voyons aussi que déterminer observationnellement la valeur de z correspondant à la domination du champ scalaire est un excellent test permettant de distinguer entre une constante cosmologique et un champ de quintessence.

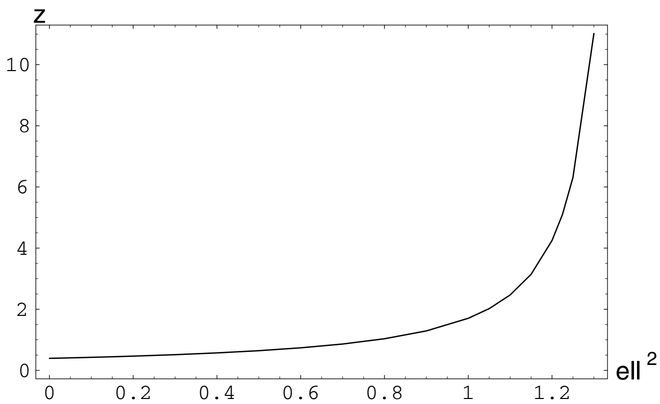

FIG. 3.4 – *Cette figure représente la courbe $\ell(z)$ vérifiant l'égalité $\Omega_{m_0}(1+z)^3 = \Omega_{\phi_0}(1+z)^{2\ell^2}$ lorsque $\Omega_{m_0} = 0.27$ et $\Omega_{\phi_0} = 0.73$. Elle diverge lorsque $\ell^2 \to 3/2\gamma$ et donne $z = 0.53$ pour $\ell^2 = 0.33$ correspondant à la valeur $w_\phi = -0.78$ déduite des observations de WMAP.*

Modèles de Bianchi courbés avec matière tel que $\Omega_m \to 0$

On a les mêmes résultats que pour la sous section précédente mais avec la restriction $\ell^2 < 1$ afin que l'Univers puisse s'isotropiser.

Modèle de Bianchi plat avec matière et champ scalaire tel que $\Omega_m \not\to 0$

Cette fois, lors de l'isotropisation, on a

- ℓ^2 tend vers une constante plus grande que $3\gamma/2$
- Les fonctions métriques tendent vers $t^{\frac{2}{3\gamma}}$
- $x \to x_0 e^{\frac{3}{2}(2-\gamma)\Omega}$, $\dot{\phi} = 2\gamma\frac{U}{U_\phi}$, $y \to (96\pi^2 R_0^6 \ell^2)^{-1}\,[3\gamma(2-\gamma)]$

Le fait que $U \propto \rho_m$ se traduit naturellement par:

$$U = \rho_\phi - p_\phi \propto \rho_m \tag{3.8}$$

Si le champ scalaire est quintessent, cela signifie que son équation d'état doit être la même que celle du fluide parfait car c'est le seul moyen d'expliquer que leurs densités d'énergie, qui s'obtiennent en écrivant leurs lois de conservation, soient proportionnelles. Vérifions le. A l'aide des limites ci-dessus, on calcule que:

$$\frac{H^2e^{6\Omega}}{144\pi^2R_0^6}\frac{3/2+\omega}{\phi^2}\dot{\phi}^2=\frac{\gamma^2}{64\pi^2R_0^6x_0^2\ell^2}e^{3\gamma\Omega}$$

et

$$U=\frac{\gamma(2-\gamma)}{64\pi^2R_0^6\ell^2x_0^2}e^{3\gamma\Omega}$$

Par conséquent, de (3.4-3.5), on obtient que $p_\phi=(\gamma-1)\rho_\phi$, ce qui est bien en accord avec (3.8) et le fait qu'à l'approche de l'isotropie les fonctions métriques tendent vers $t^{\frac{2}{3\gamma}}$ comme si il n'y avait que le fluide parfait et pas de champ scalaire. Le champ scalaire n'est donc évidement pas quintessent mais reste intéressant car il n'est pas lumineux, n'interagit pas sauf gravitationnellement avec la matière tout en simulant un accroissement de la densité d'énergie de ce dernier. Il pourrait donc jouer le rôle d'une matière noire.

3.2.3 Dynamique des anisotropies

L'isotropisation est généralement caractérisée par la quantité σ[21] telle que:

$$\sigma_{ij}=\frac{1}{2}(e^{-\beta_{si}}\frac{de^{\beta_{sj}}}{dt}+e^{-\beta_{sj}}\frac{de^{\beta_{si}}}{dt})$$

d'où l'on obtient

$$tr(\sigma^2)=6\left[(\frac{d\beta_+}{d\Omega})^2+(\frac{d\beta_-}{d\Omega})^2\right](\frac{d\Omega}{dt})^2 \tag{3.9}$$

En utilisant les équations de Hamilton et en désignant par $H_b=-\frac{d\Omega}{dt}$ la fonction de Hubble, il vient

$$X^2\doteq\frac{tr(\sigma^2)}{H_b^2}\propto x^2$$

ce qui confirme notre interprétation de la variable x en tant que variable proportionnelle au cisaillement. On considère tout d'abord un modèle sans matière ou tel que $\Omega_m\to 0$. A l'approche de l'isotropie

- pour un Univers avec sections spatiales plates, on sait que $x\to x_0e^{(3-\ell^2)\Omega}$
- pour un Univers avec courbure, on sait que $x\to x_0e^{2(1-\ell^2)\Omega}$

Par conséquent, $X(z)$ s'écrit respectivement:

$$X^2=X_0^2(1+z)^{2(3-\ell^2)}\text{ (sans courbure)}$$

$$X^2=X_0^2(1+z)^{4(1-\ell^2)}\text{ (avec courbure)}$$

X_0 étant la valeur de X à l'époque actuelle. Ces expressions ne sont valables que depuis que l'Univers subit une expansion accélérée et non au moment du CMB dont la dynamique, décélérée, ne constitue manifestement pas un état stable(on suppose donc explicitement que l'état actuel en est un!). Elles montrent que plus ℓ^2 est petit et donc l'expansion de l'univers rapide, plus l'anisotropie décroîtra vite vers notre époque ou réciproquement s'accroîtra vite vers la singularité. Cette variation de l'anisotropie dépend de la présence ou non de courbure: en sa présence, elle décroît plus vite vers les époques tardives ou, de manière inverse, elle croît plus vite en allant vers la singularité, qu'en son absence. L'expansion de l'Univers n'étant accélérée que depuis peu, l'anisotropie a décrût moins vite que la loi ci-dessus pendant la majorité de l'age de l'Univers.

Si maintenant on prend en compte la présence de matière telle que $\Omega_m\not\to 0$, nous trouvons que l'anisotropie de l'Univers se comporte comme:

$$X^2=X_0^2(1+z)^{3(2-\gamma)}$$

Cette loi ne peut évidement décrire ce qui se passe à notre époque car le cas $\Omega_m\not\to 0$ ne permet pas d'obtenir un comportement accéléré de l'expansion de l'Univers à l'approche de l'isotropie.

3.3 Modèle de Bianchi de type I avec deux champs scalaires

On considère le Lagrangien suivant:

$$S = (16\pi)^{-1} \int [R - (3/2+\omega)\phi^{,\mu}\phi_{,\mu}\phi^{-2} - (3/2+\mu)\psi^{,\mu}\psi_{,\mu}\psi^{-2} - U \\ +16\pi c^4 L_m]\sqrt{-g}d^4x$$

avec les fonctions de couplage de Brans-Dicke $\omega(\phi,\psi)$, $\mu(\phi,\psi)$ et un potentiel $U(\phi,\psi)$. De plus on rappelle les définitions des fonctions du champ scalaire $\ell_{\phi_1} = \phi U_\phi U^{-1}(3+2\omega)^{-1/2}$, $\ell_{\psi_1} = \psi U_\psi U^{-1}(3+2\mu)^{-1/2}$, $\ell_{\phi_2} = \phi\mu_\phi(3+2\mu)^{-1}(3+2\omega)^{-1/2}$ et $\ell_{\psi_2} = \psi\omega_\psi(3+2\omega)^{-1}(3+2\mu)^{-1/2}$. Nous considérons toujours que $3+2\omega > 0$, $3+2\mu > 0$ et $U > 0$. Comme précédemment, il nous faut déterminer la pression et la densité d'énergie de chacun des champs scalaires tels que leurs tenseurs d'énergie-impulsion prennent, si possible, la forme de celui d'un fluide parfait. La somme des tenseurs énergie-impulsion des deux champs scalaires s'écrit:

$$T_{\alpha\beta} = \frac{1}{2}\frac{3+2\omega}{\phi^2}\phi_{,\alpha}\phi_{,\beta} + \frac{1}{2}\frac{3+2\mu}{\psi^2}\psi_{,\alpha}\psi_{,\beta} - \frac{1}{4}\frac{3+2\omega}{\phi^2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}g_{\alpha\beta} - \\ \frac{1}{4}\frac{3+2\mu}{\psi^2}g^{\mu\nu}\psi_{,\mu}\psi_{,\nu}g_{\alpha\beta} - \frac{1}{2}Ug_{\alpha\beta}$$

et on définit les vecteurs du genre temps relatifs aux deux champs scalaires:

$$u_\alpha = \frac{\phi_{,\alpha}}{\sqrt{-g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}}}$$

$$v_\alpha = \frac{\psi_{,\alpha}}{\sqrt{-g^{\mu\nu}\psi_{,\mu}\psi_{,\nu}}}$$

On en déduit alors que

$$\begin{aligned} p_\phi + p_\psi &= \frac{1}{4}\frac{3+2\omega}{\phi^2}\phi'^2 + \frac{1}{4}\frac{3+2\mu}{\psi^2}\psi'^2 - \frac{1}{2}U \\ \rho_\phi + p_\phi &= \frac{1}{2}\frac{3+2\omega}{\phi^2}\phi'^2 \\ \rho_\psi + p_\psi &= \frac{1}{2}\frac{3+2\mu}{\psi^2}\psi'^2 \end{aligned}$$

soient trois équations pour quatre inconnues, d'où une forme d'indétermination sur les pressions et densités d'énergie des champs scalaires. Dans ce qui suit on examine le processus d'isotropisation de chacune des deux classes de théories définies par $(\omega(\phi),\mu(\psi),U(\phi,\psi))$ et $(\omega(\phi,\psi),\mu(\psi),U(\psi))$.

3.3.1 $\omega = \omega(\phi)$, $\mu = \mu(\psi)$, $U = U(\phi,\psi)$ et $\Omega_m \to 0$

A l'approche de l'isotropie, nous rappelons les résultats suivants:

- $\ell_{\phi_1}^2 + \ell_{\psi_1}^2$ tend vers une constante plus petite que 3 (sans matière) ou $3\gamma/2$ (avec matière)
- Si cette constante est non nulle, les fonctions métriques tendent vers $t^{(\ell_{\phi_1}^2+\ell_{\psi_1}^2)^{-1}}$ et le potentiel vers t^{-2}. Sinon l'Univers tend vers un modèle de De Sitter et le potentiel vers une constante

Nous ne rappelons pas les limites asymptotiques de x, ϕ et y qui seraient trop longues à énoncer et renvoyons le lecteur intéressé vers le chapitre 1 de cette partie. Pour surmonter l'indétermination, on suppose que:

$$p_\phi = \frac{1}{4}\frac{3+2\omega}{\phi^2}\phi'^2 - a\frac{1}{2}U$$

d'où l'on déduit que

$$p_\psi = \frac{1}{4}\frac{3+2\mu}{\psi^2}\psi'^2 - (1-a)\frac{1}{2}U$$

$$\rho_\phi = \frac{1}{4}\frac{3+2\omega}{\phi^2}\phi'^2 + a\frac{1}{2}U$$

$$\rho_\psi = \frac{1}{4}\frac{3+2\mu}{\psi^2}\psi'^2 + (1-a)\frac{1}{2}U$$

où a est une constante qu'il nous faut calculer afin de déterminer complètement cette solution. Or à l'approche de l'isotropie, nous avons:

$$\frac{1}{4}\frac{3+2\omega}{\phi^2}\phi'^2 = \frac{H^2 e^{6\Omega}}{288\pi^2 R_0^6}\frac{3/2+\omega}{\phi^2}\dot{\phi}^2 = \frac{e^{2(\ell_{\phi_1^2}+\ell_{\psi_1^2})\Omega}}{144\pi^2 R_0^6 x_0^2}\ell^2_{\phi_1}$$
$$\frac{1}{4}\frac{3+2\mu}{\psi^2}\psi'^2 = \frac{H^2 e^{6\Omega}}{288\pi^2 R_0^6}\frac{3/2+\mu}{\psi^2}\dot{\psi}^2 = \frac{e^{2(\ell_{\phi_1^2}+\ell_{\psi_1^2})\Omega}}{144\pi^2 R_0^6 x_0^2}\ell^2_{\psi_1}$$
$$\frac{U}{2} = \frac{3-\ell_{\phi_1^2}-\ell_{\psi_1^2}}{144\pi^2 R_0^6 x_0^2}e^{2(\ell_{\phi_1^2}+\ell_{\psi_1^2})\Omega}$$

De ces expressions, on déduit que si ℓ_{ϕ_1} tend vers zéro mais pas ℓ_{ψ_1}, la contribution cinétique du champ scalaire ϕ dans le tenseur d'énergie-impulsion peut être négligée. L'indétermination est alors levée: il n'y a plus besoin de faire intervenir de constante a. Tout se passe comme si nous n'avions plus qu'un seul champ scalaire et on retrouve les mêmes résultats que dans la section précédente en l'absence de matière ou avec $\Omega_m \to 0$ et tel que $\ell \to \ell_{\psi_1}$. Ceci est confirmé par le fait qu'alors les fonctions métriques tendront asymptotiquement vers $t^{\ell_{\psi_1}^{-2}}$. On peut évidement faire le même raisonnement pour ℓ_{ψ_1}.
Si ℓ_{ϕ_1} et ℓ_{ψ_1} tendent vers des constantes non nulles, tous les termes entrant dans l'expression du tenseur d'énergie-impulsion varient comme $e^{2(\ell^2_{\phi_1}+\ell^2_{\psi_1})\Omega}$. Par conséquent, il en est de même des densités d'énergie et des pressions des deux champs scalaires dont les indices barotropiques doivent tendre vers une constante commune w telle que $w = w_\phi = w_\psi$. Or ces indices s'expriment comme:

$$\frac{p_\phi}{\rho_\phi} = w_\phi = \frac{\ell_{\phi_1^2} - a(3-\ell_{\phi_1^2}-\ell_{\psi_1^2})}{\ell_{\phi_1^2} + a(3-\ell_{\phi_1^2}-\ell_{\psi_1^2})}$$
$$\frac{p_\psi}{\rho_\psi} = w_\psi = \frac{\ell_{\psi_1^2} - (1-a)(3-\ell_{\phi_1^2}-\ell_{\psi_1^2})}{\ell_{\psi_1^2} + (1-a)(3-\ell_{\phi_1^2}-\ell_{\psi_1^2})}$$

En les égalant, on obtient:

$$a = \frac{\ell_{\phi_1^2}}{\ell_{\phi_1^2}+\ell_{\psi_1^2}}$$
$$w_\phi = w_\psi = \frac{2}{3}(\ell_{\phi_1^2}+\ell_{\psi_1^2}) - 1$$

w_ϕ et w_ψ tendent asymptotiquement vers une constante lors de l'isotropisation, cet indice barotropique généralisant celui trouvé en présence d'un unique champ scalaire dans la section précédente et étant en accord avec la convergence des fonctions métriques vers $t^{(\ell^2_{\phi_1}+\ell^2_{\psi_1})^{-1}}$ à l'approche de l'isotropie. L'expression des densités d'énergie des deux champs scalaires en fonction du redshift donne:

$$\rho_\phi = \rho_{\phi_0}(1+z)^{2(\ell^2_{\phi_1}+\ell^2_{\psi_1})}$$
$$\rho_\psi = \rho_{\psi_0}(1+z)^{2(\ell^2_{\phi_1}+\ell^2_{\psi_1})}$$

et donc la fonction de Hubble s'écrit comme:

$$\begin{aligned} H^2 &= H_0^2\left[\Omega_{m_0}(1+z)^3 + \Omega_{\phi_0}(1+z)^{2(\ell^2_{\phi_1}+\ell^2_{\psi_1})} + \Omega_{\psi_0}(1+z)^{2(\ell^2_{\phi_1}+\ell^2_{\psi_1})}\right] \\ &= H_0^2\left[\Omega_{m_0}(1+z)^3 + (\Omega_{\phi_0}+\Omega_{\psi_0})(1+z)^{2(\ell^2_{\phi_1}+\ell^2_{\psi_1})}\right] \\ &= H_0^2 E(z) \end{aligned}$$

Ces résultats indiquent qu'asymptotiquement lorsque l'Univers s'isotropise, on ne peut distinguer entre la présence d'un ou de plusieurs champs scalaires. Pour les observations, tout se passe comme si nous avions un unique champ scalaire dont l'amplitude de la densité d'énergie serait $\rho_{\phi_0}+\rho_{\psi_0}$ avec un index barotropique $w = \frac{2}{3}(\ell_{\phi_1^2}+\ell_{\psi_1^2}) - 1$, c'est-à-dire comme si l'on avait remplacé le ℓ^2 de la section précédente par $\ell_{\phi_1^2}+\ell_{\psi_1^2}$. On peut cependant montrer que le rapport constant des deux densités d'énergies (et donc des deux pressions) est:

$$\frac{\rho_\phi}{\rho_\psi} = \frac{3\ell^2_{\phi_1}}{3\ell^2_{\psi_1} + (\ell^4_{\phi_1} - \ell^4_{\psi_1})}$$

3.3.2 Isotropisation quintessente lorsque $\omega = \omega(\phi,\psi)$, $\mu = \mu(\psi)$, $U = U(\psi)$ et $\Omega_m \to 0$

Il existe deux points d'équilibre E_1 et E_2 correspondant à un état isotrope stable pour l'Univers.

Cas du point d'équilibre E_1

A l'approche de l'isotropie, nous avons que:

- $\ell^2_{\psi_1}$ tend vers une constante plus petite que 3 (sans matière) ou $3\gamma/2$ (avec matière)
- Si cette constante est non nulle, les fonctions métriques tendent vers $t^{\ell^{-2}_{\psi_1}}$ et le potentiel vers t^{-2}. Sinon l'Univers tend vers un modèle de De Sitter et le potentiel vers une constante

Pour les mêmes raisons que dans la section précédente, nous n'avons pas mis toutes les conditions nécessaires et les limites asymptotiques de toutes les quantités à l'approche de l'isotropie. On calcule les termes apparaissant dans le tenseur d'énergie-impulsion des champs scalaires:

$$\frac{1}{4}\frac{3+2\omega}{\phi^2}\phi'^2 = \frac{H^2 e^{6\Omega}}{288\pi^2 R_0^6}\frac{3/2+\omega}{\phi^2}\dot{\phi}^2 = \frac{e^{6\Omega - 4\int \ell_{\psi_1}\ell_{\psi_2} d\Omega}}{4\pi^2 R_0^6 x_0^2}$$

$$\frac{1}{4}\frac{3+2\mu}{\psi^2}\psi'^2 = \frac{H^2 e^{6\Omega}}{288\pi^2 R_0^6}\frac{3/2+\mu}{\psi^2}\dot{\psi}^2 = \frac{e^{2\ell^2_{\psi_1}\Omega}}{144\pi^2 R_0^6 x_0^2}\ell^2_{\psi_1}$$

$$\frac{U}{2} = \frac{3-\ell_{\psi_1^2}}{144\pi^2 R_0^6 x_0^2} e^{2\ell^2_{\psi_1}\Omega}$$

Or, l'isotropie nécessite que $(3-\ell^2_{\psi_1})\Omega - 2\int \ell_{\psi_1}\ell_{\psi_2} d\Omega \to -\infty < 0$, ce qui implique que $6\Omega - 4\int \ell_{\psi_1}\ell_{\psi_2} d\Omega < 2\ell^2_{\psi_1}\Omega < 0$. Il s'ensuit qu'asymptotiquement la partie cinétique du champ scalaire ϕ entrant dans la composition du tenseur d'énergie-impulsion est négligeable face au potentiel. Lorsque ℓ_{ψ_1} tend vers une quantité non nulle, le tenseur d'énergie-impulsion peut donc être approximé par:

$$T_{\alpha\beta} = \frac{1}{2}\frac{3+2\mu}{\psi^2}\psi_{,\alpha}\psi_{,\beta} - \frac{1}{4}\frac{3+2\mu}{\psi^2}g^{\mu\nu}\psi_{,\mu}\psi_{,\nu}g_{\alpha\beta} - \frac{1}{2}U g_{\alpha\beta}$$

Il correspond alors à celui trouvé en présence d'un unique champ scalaire et est en accord avec la forme asymptotique des fonctions métriques lors de l'isotropisation, $t^{\ell^{-2}_{\psi_1}}$, montrant que ϕ n'a pas une influence dynamique détectable. Lorsque ℓ_{ψ_1} tend vers zéro, seul le potentiel n'est pas négligeable dans le tenseur d'énergie-impulsion et donc l'indice barotropique tend vers -1 en accord avec le fait que l'Univers tend alors vers un modèle de De Sitter et le potentiel vers une constante cosmologique.

Cas du point d'équilibre E_2

A l'approche de l'isotropie, nous avons entre autre que:

- $0 < 2\ell_{\psi_2}(\ell_{\psi_1} + 2\ell_{\psi_2})^{-1} < 1$ en l'absence de matière et $1 - \gamma/2 < 2\ell_{\psi_2}(\ell_{\psi_1} + 2\ell_{\psi_2})^{-1} < 1$ en présence de matière.
- $\ell_{\psi_1} + 2\ell_{\psi_2} \neq 0$ et $\ell_{\psi_1}(\ell_{\psi_1} + 2\ell_{\psi_2}) > 3$
- Si $\ell_{\psi_1}(\ell_{\psi_1}+2\ell_{\psi_2})^{-1}$ tend vers une constante non nulle, les fonctions métriques tendent vers $t^{(\ell_{\psi_1}+2\ell_{\psi_2})(3\ell_{\psi_1})^{-1}}$ et le potentiel vers t^{-2}. Sinon l'Univers tend vers un modèle de De Sitter et le potentiel vers une constante

De plus, on calcule que:

$$\frac{1}{4}\frac{3+2\omega}{\phi^2}\phi'^2 = \frac{H^2 e^{6\Omega}}{288\pi^2 R_0^6}\frac{3/2+\omega}{\phi^2}\dot{\phi}^2 = \frac{e^{\frac{6\ell_{\psi_1}}{\ell_{\psi_1}+2\ell_{\psi_2}}\Omega}}{4\pi^2 R_0^6 x_0^2}\frac{\ell^2_{\psi_1} + 2\ell_{\psi_1}\ell_{\psi_2} - 3}{12(\ell_{\psi_1}+2\ell_{\psi_2})^2}$$

$$= \frac{e^{\frac{6\ell_{\psi_1}}{\ell_{\psi_1}+2\ell_{\psi_2}}\Omega}}{48\pi^2 R_0^6 x_0^2}\left(\frac{\ell_{\psi_1}}{\ell_{\psi_1}+2\ell_{\psi_2}} - \frac{3}{(\ell_{\psi_1}+2\ell_{\psi_2})^2}\right)$$

$$\frac{1}{4}\frac{3+2\mu}{\psi^2}\psi'^2 = \frac{H^2 e^{6\Omega}}{288\pi^2 R_0^6}\frac{3/2+\mu}{\psi^2}\dot{\psi}^2 = \frac{e^{\frac{6\ell_{\psi_1}}{\ell_{\psi_1}+2\ell_{\psi_2}}\Omega}}{4\pi^2 R_0^6 x_0^2}\frac{1}{4(\ell_{\psi_1}+2\ell_{\psi_2})^2}$$

$$\frac{U}{2} = \frac{e^{\frac{6\ell_{\psi_1}}{\ell_{\psi_1}+2\ell_{\psi_2}}\Omega}}{24\pi^2 R_0^6 x_0^2}\frac{\ell_{\psi_2}}{\ell_{\psi_1}+2\ell_{\psi_2}} = -\frac{e^{\frac{6\ell_{\psi_1}}{\ell_{\psi_1}+2\ell_{\psi_2}}\Omega}}{48\pi^2 R_0^6 x_0^2}(\frac{\ell_{\psi_1}}{\ell_{\psi_1}+2\ell_{\psi_2}} - 1)$$

Ici tout dépend du comportement des deux fonctions $F_1 = \frac{\ell_{\psi_1}}{\ell_{\psi_1}+2\ell_{\psi_2}}$ et $F_2 = \frac{3}{\ell_{\psi_1}+2\ell_{\psi_2}}$ sachant que ℓ_{ψ_1} et ℓ_{ψ_2} peuvent diverger à l'approche de l'isotropie mais pas F_1 et F_2 car les points d'équilibre sont finis.
Si elles tendent toutes les deux vers des constantes non nulles, les densités d'énergie des deux champs se comportent de la même manière. On peut donc supposer comme précédemment qu'à l'approche de l'isotropie leurs indices barotropiques doivent être les mêmes. On procède donc de la même manière pour trouver la valeur de a et il vient:

$$a = \frac{\ell_{\psi_1}^2 + 2\ell_{\psi_1}\ell_{\psi_2} - 3}{\ell_{\psi_1}(2\ell_{\psi_2} + \ell_{\psi_1})}$$

et pour l'indice barotropique:

$$w = w_\phi = w_\phi = \frac{2\ell_{\psi_1}}{\ell_{\psi_1} + 2\ell_{\psi_2}} - 1$$

A nouveau la présence des deux champs scalaires mime la présence d'un seul d'entre eux avec un indice barotropique $w \in [-1,1]$ et tel que le paramètre ℓ^2 soit remplacé par $\frac{3\ell_{\psi_1}}{\ell_{\psi_1}+2\ell_{\psi_2}}$. Observationnellement, on ne peut donc les distinguer lorsque l'Univers s'isotropise. Notons la différence avec le cas précédent: pour le point E_1, l'un des champs scalaires est asymptotiquement négligeable alors que pour le point E_2, les deux champs scalaires se comportent de la même manière. Cette fois le rapport entre les deux densités d'énergie est:

$$\frac{\rho_\phi}{\rho_\psi} = \frac{1}{3}(\ell_{\psi_1}^2 + 2\ell_{\psi_1}\ell_{\psi_2} - 3)$$

Si $F_1 \to 0$, une des conditions nécessaire à l'isotropie étant que $\ell_{\psi_1}(\ell_{\psi_1} + 2\ell_{\psi_2}) > 3$, cela implique que $\ell_{\psi_1} + 2\ell_{\psi_2}$ diverge et donc que $F_2 \to 0$. Dans ce cas l'Univers tend vers un modèle de De Sitter et w vers -1.
Si $F_2 \to 0$ et F_1 vers une constante non nulle, le terme cinétique du champ scalaire ψ devient négligeable mais on retrouve les mêmes résultats pour w que lorsque les deux fonctions tendent vers des constantes non nulles.

3.3.3 Cas pour lequel $\Omega_m \not\to 0$

Asymptotiquement, les fonctions métriques convergent vers la fonction $t^{\frac{2}{3\gamma}}$ et donc la somme des densités d'énergie des champs scalaires s'équilibre avec celle du fluide parfait comme en présence d'un unique champ scalaire.

3.3.4 Dynamique des anisotropies

En ce qui concerne les théories tenseur-scalaires de la section 3.3.1, on a:

$$X^2 = X_0^2(1+z)^{2(3-\ell_{\phi_1}^2-\ell_{\psi_1}^2)}$$

Pour les théories définies dans la section 3.3.2, on a pour le point d'équilibre E_1:

$$X^2 = X_0^2(1+z)^{2(3-\ell_{\psi_1}^2)}$$

et pour le point d'équilibre E_2:

$$X^2 = X_0^2(1+z)^{2(3-\frac{3\ell_{\psi_1}}{\ell_{\psi_1}+2\ell_{\psi_2}})}$$

Là encore, la manière dont les anisotropies évoluent est indiscernable de celle trouvée en la présence d'un unique champ scalaire ϕ lorsque l'on remplace respectivement ℓ^2 par $\ell_{\phi_1}^2 + \ell_{\psi_1}^2$, $\ell_{\psi_1}^2$ et $\frac{3\ell_{\psi_1}}{\ell_{\psi_1}+2\ell_{\psi_2}}$.

3.4 Avec champ scalaire non minimalement couplé

Ici, il ne semble pas possible de déterminer le ou les vecteurs de genre temps permettant d'écrire le tenseur d'énergie-impulsion comme celui d'un fluide parfait. Afin de déterminer la densité et la pression du champ scalaire, on suit donc la méthode utilisée par exemple dans [132, 133]. On définit l'action propre au champ scalaire comme

$$S_\phi = \int \left[(G^{-1}-1)R - (\frac{3}{2}+\omega)\phi^{-2}g_{\mu\nu}\phi^\mu\phi^\nu - U\right] d^4x\sqrt{g}$$

On la varie par rapport aux fonctions métriques afin d'obtenir le tenseur d'énergie-impulsion de la théorie tenseur-scalaire définie par $S = S_R + S_\phi$ où $S_R = \int Rd^4x\sqrt{g}$. Il vient:

$$T_{\alpha\beta} = \frac{1}{2}\frac{3+2\omega}{\phi^2}\phi_{,\alpha}\phi_{,\beta} - \frac{1}{4}\frac{3+2\omega}{\phi^2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}g_{\alpha\beta} + (G^{-1}-1)_{,\alpha;\beta} - g_{\alpha\beta}\Box(G^{-1}-1) - \frac{1}{2}Ug_{\alpha\beta} - (G^{-1}-1)G_{\alpha\beta}$$

où $G_{\alpha\beta}$ est le tenseur d'Einstein. On défini alors la densité et la pression du champ scalaire comme:

$$\rho_\phi = T_{00}$$

$$p_\phi = \frac{1}{3}(e^{-2\alpha}T_{11} + e^{-2\beta}T_{22} + e^{-2\gamma}T_{33})$$

ce qui donne lorsque l'Univers s'isotropise:

$$\rho_\phi = \frac{1}{4}\frac{3+2\omega}{\phi^2}\frac{d\phi}{dt}^2 + \frac{1}{2}U - 3\frac{d\Omega}{dt}\left[-\frac{d(G^{-1}-1)}{dt} + (G^{-1}-1)\frac{d\Omega}{dt}\right]$$

$$p_\phi = \frac{1}{4}\frac{3+2\omega}{\phi^2}\frac{d\phi}{dt}^2 - \frac{1}{2}U + \frac{d^2(G^{-1}-1)}{dt^2} - 2\frac{d(G^{-1}-1)}{dt}\frac{d\Omega}{dt} - (G^{-1}-1)(2\frac{d^2\Omega}{dt^2} - 3\frac{d\Omega}{dt}^2)$$

Ces expressions généralisent pour tout G les formes de la densité d'énergie et de la pression du champ scalaire qui avaient été calculées dans [132] pour le cas spécial $(G^{-1}-1) = -\xi\phi^2$. Lorsque l'on considère une fonction de gravitation tendant vers la constante 1 représentant sa valeur actuelle, on retrouve sans surprise les mêmes résultats que dans la section 3.2 où le champ scalaire est minimalement couplé. Dans le chapitre 1, nous avons étudié l'isotropisation de cette théorie en nous plaçant dans le référentiel d'Einstein dans lequel le champ scalaire est minimalement couplé à la courbure mais pas à la matière. Si G au lieu de tendre vers une constante tend vers une puissance du temps propre du référentiel d'Einstein $\bar{t}^m$ comme cela peut être le cas, et comme asymptotiquement $\Omega = \Omega_{BD} + 1/2\ln G^{-1}$ tend vers un $\ln\bar{t}$ et $\bar{U} \to \bar{t}^{-2}$ lors de l'isotropisation, nous trouvons que la pression et la densité du champ scalaire tendent vers un polynôme de la forme $\bar{t}^{-m-2} + \bar{t}^{-2m-2}$ et que par conséquent le champ scalaire peut à nouveau être quintessent en fonction de la valeur de m.
Nous avons testé plusieurs autres fonctions de G^{-1} pour la même forme de Ω prédite lors de l'isotropisation. De manière générale, la quintessence peut exister dans les cas où $G \to m\ln\bar{t}^s$, $G \to e^{-m\bar{t}}$ ou $G \to m\bar{t}^p\ln\bar{t}^s$ sauf éventuellement des choix spéciaux des paramètres (n,m,p,s). Bien que nous n'ayons pas trouvé de démonstration rigoureuse permettant d'affirmer que le champ scalaire non minimalement couplé devient quintessent lors d'une isotropisation de classe 1, il y a de fortes présomptions pour que cela soit généralement le cas.

3.5 Discussion

Dans ce chapitre nous avons examiné si le champ scalaire est quintessent lorsque l'Univers s'isotropise. Nous avons alors cherché le redshift correspondant à la domination du champ scalaire sur la matière et la forme des anisotropies. Nous sommes principalement intéressés par les états isotropes tels que l'expansion de l'Univers soit asymptotiquement accélérée, c'est-à-dire, en présence de matière, tel que $\Omega_m \to 0$.
Lorsque nous considérons un champ scalaire, nous avons trouvé que la constante vers laquelle tend la fonction ℓ lors de l'isotropisation peut être asymptotiquement interprétée comme étant l'indice barotropique de l'équation d'état caractérisant le champ scalaire. Celui-ci peut alors être quintessent à l'approche de l'isotropie pour le modèle de Bianchi de type I vide ou tel que $\Omega_m \to 0$ si $\ell^2 < 3/2$. En revanche, en présence

de courbure c'est systématiquement le cas car l'isotopisation impose que $\ell^2 < 1$. La valeur de ℓ déduite des observations de WMAP est $\ell^2 < 0.33$ avec $\Omega_{\phi 0} = 0.73$. Elle correspond à la présence d'un fluide quintessent dont la densité d'énergie évolue comme $\rho_\phi = \rho_{\phi_0}(1+z)^{2\ell^2}$ avec $\rho_{\phi_0} = 6.894.10^{-30} g/cm^3$ et qui est devenue supérieure à la densité d'énergie du fluide parfait en $z < 0.53$. Nous avons également déterminé l'évolution asymptotique en fonction du redshift de la partie anisotrope de la métrique, $tr(\sigma^2)H^{-2}(z)$ et constaté que l'anisotropie se dissipait plus vite en présence de courbure.
En présence de deux champs scalaires, les choses sont plus compliquées. Nous avons étudié deux classes de théories tenseur-scalaires qui se différencient par la dépendance de leurs fonctions de Brans-Dicke et potentiel vis à vis des deux champs scalaires ϕ et ψ. Deux cas se présentent. Dans le premier, tous les termes rentrant dans la composition du tenseur énergie-impulsion des deux champs scalaires se comportent de la même manière à l'approche de l'isotropie. On peut alors supposer que tout se passe comme si les densités d'énergies des deux champs scalaires étaient conservées séparément et que leurs indexes barotropiques étaient les mêmes. On peut définir les pressions et densités de chaque champ scalaire et calculer leur indice barotropique commun. Dans le second cas, l'un des termes cinétiques d'un des champs scalaires, disons ψ, apparaissant dans le tenseur d'énergie-impulsion est négligeable. Ce tenseur prend alors la même forme qu'en présence d'un seul champ scalaire ϕ et la forme asymptotique des fonctions métrique est bien en accord avec la domination de ce champ sur l'autre. Dans les deux cas, la présence d'un second champ scalaire est observationnellement indiscernable. Tout se passe comme si il n'en existait qu'un seul.
Lorsque nous considérons la présence d'un champ scalaire non minimalement couplé, comme l'isotropisation n'impose pas de formes asymptotiques particulières à la fonction de gravitation G, il n'est pas possible de montrer de manière générale (c'est-à-dire pour n'importe quelle forme de G) que le champ scalaire peut être quintessent à l'approche de l'isotropie. Cependant, cela semble être le cas pour de nombreuses formes de $G(t)$
De manière générale, on constate qu'une isotropisation de classe 1 pour la classe de théories tenseur-scalaires étudiée ici est compatible avec la quintessence. En présence de courbure celle-ci apparaît même systématiquement. Cette isotropisation quintessente pourrait ne pas se produire si l'isotropisation était de classe 2 ou 3 pour lesquelles ℓ peut diverger ou osciller.

Chapitre 4

Matière noire(1 article)

De nombreuses observations montrent que 90 à 99% de la matière de l'Univers est invisible et donc sous la forme de ce que l'on appelle de la matière noire. On en ignore la nature mais il semble qu'une fraction de celle ci soit composée de matière baryonique, la partie restante pouvant être formée par exemple de particules élémentaires comme des neutrinos ou des WIMP.
Comment mesure t'on la présence de matière noire? Prenons par exemple le cas d'un amas de galaxies. On peut mesurer sa masse soit en additionnant les masses individuelles des galaxies le composant, soit en estimant leurs vitesses particulières et en se servant de la relation $V^2R^{-1} = GMR^{-2}$. En procédant ainsi sur l'amas de Coma, cette dernière méthode nous donne une masse 10 à 100 fois plus grande que celle obtenue avec la première. En ce qui concerne les galaxies, la détection de la matière noire peut se faire à l'aide de leur vitesse de rotation. En effet, celle-ci devrait croître au niveau du centre galactique où l'essentielle de la matière semble se trouver puis, selon les lois de Képler, elle devrait décroître dans les régions externes de ces objets. Or pour de nombreuses galaxies, la vitesse de rotation ne décroît pas, indiquant le présence d'une importante quantité de matière invisible [1].
C'est à la nature de cette matière dans les galaxies que ce chapitre est consacré. Comme nous l'avons indiqué plus haut, il peut s'agir de particules élémentaires comme des neutrinos ou des WIMPs. Dans le cas des neutrinos qui sont très légers, ces particules se déplacent très vite (Hot Dark Matter) et doivent donc avoir parcouru de grandes distances. Les structures qu'une telle matière est capable de former doivent donc s'étaler sur de grandes échelles sous forme de murs ou de filaments. Au contraire dans le cas des WIMPs qui sont des particules massives et relativement lentes (Cold Dark Matter), les structures formées devraient être plus petites, à l'échelle des galaxies. Notons que d'autres explications qu'une forme inconnue de matière sont possibles pour expliquer la dynamique des amas et des galaxies. Ainsi la théorie MOND [138, 139, 140] (MOdified Newton Dynamics) inventée par Mordehai Milgrom ne fait pas intervenir de matière mais plutôt une modification des lois de la gravité.
Suivant une idée de Matos et al[136], nous allons considérer que l'applatissement des courbes de rotation des galaxies, principalement spirales, est due à la présence d'un ou plusieurs champs scalaires. La forme de la métrique qui en résulte, initialement supposée sphérique et statique, est alors déduite ainsi qu'une contrainte sur les champs scalaires sous forme d'une limite similaire à celle trouvée pour l'isotropisation des modèles de Bianchi. Ce dernier point est, de notre point de vue, particulièrement important car il montre que la sélection des propriétés des champs scalaires qui résulte de l'exigence d'isotropisation pourrait se traduire par des effets dynamiques au niveau galactique, établissant ainsi un lien entre l'énergie sombre qui se manifeste par la quintessence et la matière noire.

1. Les effets de la mati`ere noire dans les amas et les galaxies sont tr`es bien illustr´ees `a l'aide d'applets java sur http://www.astro.queensu.ca/ dursi/dm-tutorial/dm1.html

Scalar fi elds properties for flat galactic rotation curves.

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Abstract

The whole class of minimally coupled and massive scalar fields which may be responsible for flattening of galactic rotation curves is found. An interesting relation with a class of scalar-tensor theories able to isotropise anisotropic models of Universe is shown. The resulting metric is found and its stability and scalar field properties are tested with respect to the presence of a second scalar field or a small perturbation of the rotation velocity at galactic outer radii.

keywords: galactic rotation curves – scalar fields – dark matter

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4.1 Introduction

One of the most fascinating cosmological problems is dark matter one: 99 percent of the Universe energy density would be hidden. A good indication of dark matter presence is given by galactic rotation curves which disagree with Kepler laws [134]. Particularly, spiral galaxy rotation curves seem flattened at large radii. One possible explanation is that they are made of a luminous disk whose density exponentially decreased to adjust to a dark halo whose distribution evolves as r^{-2} [135]
Dark matter nature is unknown today. From WMAP observations, we know that the matter of which we are made represents 4% of the Universe content, 23% is made of cold dark matter and 73% of dark energy. These exotic types of matter could be represented by scalar fields [136], which are predicted by unification theories [113]. Starting from this assumption, we will study how they could be responsible for the observed flattened rotation curves. Dark matter is not the only way to explain them [137]. Hence, Milgrom [138, 139, 140], Sanders [141] and others have proposed to modify newtonian theory(MOND) for galaxies outer radii and in [142], ad hoc magnetic field are considered.

Hence, the physical framework of this paper will be the scalar tensor theories. Unification theories and particularly supersymmetry predict the existence of scalar fields, which thus deserve be taken into account in cosmology. Most of time, only their cosmological consequences are analysed: quintessence phenomenon [143, 144], isotropisation [105] or inflation for instance. However, they could also be present at galactic scale. In this work, we are going to assume that the dynamics of galaxies at outer radii is described by a scalar tensor theory with a dust perfect fluid, neglecting the radiation. The scalar field ϕ will be minimally coupled, massive with a Brans-Dicke function representing its coupling with the metric. It is equivalent to consider that, at galactic scale, the gravitational function is a constant but the potential U and the Brans-Dicke coupling function ω vary with ϕ.

Concerning the geometrical framework, we will consider a spherical and static metric. These are reasonable assumptions since, generally, a galaxy has a rotation axis around which turn the stars with a velocity much smaller than light speed. We thus neglect dragging effects, justifying a static metric [145]. We will be interested by galactic regions where rotation curves flatten and where most of the dark matter should be present, i.e. the galaxies outer radii. Indeed internal regions need few or not dark matter to explain their dynamics.

Our goal will be to study the form of the metric and the scalar field properties explaining why the galactic rotation curves are flattened. A similar work has been done in [146] and [147]. In the first paper, a massive scalar tensor theory is studied with a fixed ω but an unknown U. After having found the metric compatible with flat rotation curves, it is shown that the only potential able to reproduce such a dynamics should have an exponential form, $U = e^{k\phi}$. In the second paper, with the same form for ω, two massive scalar fields and a perfect fluid are considered. One of the potentials is assumed to have an exponential form and similar results are found. In the present paper, we are going to consider a single massive scalar field with a perfect fluid but both ω and U will be unknown functions of ϕ. Then we will look for the metric and relations between ω and U allowing to get the observed flat rotation curves, thus generalising the results of [146] to a larger class of scalar tensor theories. Moreover, we will test the stability of our results by considering a small perturbation of the rotation velocity or/and an additional scalar field.
It is important to note that other types of rotation curves exist, such as decreasing rotation curves found by

Casertano and van Gorkom [148]. Moreover, it seems that bright compact galaxy rotation curves are slightly decreasing whereas low luminosity ones tend to be increasing, indicating that they have more dark matter. However in this work we will only take into account asymptotically flat rotation curves. Indeed, a large number of them seems to be well approximated by an Universal Rotation Curve [149] whose formulation, adapted to spiral galaxies, tends to a constant at late times. It shows how rotation curves depend on galaxy luminosity: increasing or decreasing rotation curves would respectively correspond to low or high galactic luminosity but should tend to a constant at outer radii. It seems to be confirmed by Swaters [150] who has examined a large number of dwarf galaxies and found that their rotation curves flattened over 2 disk scale lengths. MOND theories or the presence of electromagnetic fields can also predict this type of curves. Consequently, although all the rotation curves do not flatten, this type of behaviour is sufficiently observed or predicted to justify a particular attention.

The plane of this paper is the following. In section 2, we look for metric form and scalar field properties allowing to get flattened rotation curves. In section 3, we discuss about these results.

4.2 Metric and scalar field mathematical properties

This section is divided in two parts. In the first one, we consider the presence of a single scalar field. We look for the metric and properties of the unknown functions ω and U such that the rotation curves be flattened at galaxy outer radii. In the second one, we test our results stability by adding a second scalar field. We will use a static and spherical metric written as:

$$ds^2 = -e^{2\phi}dt^2 + e^{2\Lambda}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{4.1}$$

ϕ and Λ being some functions of r.

4.2.1 With a single scalar field

The action for a minimally coupled and massive scalar field with a perfect fluid is given by:

$$S = \int (R - \frac{\omega}{\psi}\psi_{,\mu}\psi^{,\mu} - U + \frac{16\pi}{c^4}L_m)\sqrt{-g}d^4x \tag{4.2}$$

R is the Ricci scalar, ψ the scalar field, $\omega(\psi)$ the Brans-Dicke coupling function and $U(\psi)$ the potential. L_m is the Lagrangian describing a dust perfect fluid whose impulsion-energy tensor writes $T^{\alpha\beta} = \rho u^\alpha u^\beta$ with u^α the 4-velocity and ρ the density of the dust fluid. We get the field equations and Klein-Gordon equation by varying the action with respect to the metric functions and scalar field:

$$r^{-2}\left[r(1-e^{-2\Lambda})\right]' = \frac{\omega}{2\psi}\psi'^2 e^{-2\Lambda} + 1/2U + \rho \tag{4.3}$$

$$-r^{-2}(1-e^{-2\Lambda}) + 2r^{-1}\phi' e^{-2\Lambda} = \frac{\omega}{2\psi}\psi'^2 e^{-2\Lambda} - 1/2U \tag{4.4}$$

$$e^{-2\Lambda}(\phi'' + \phi'^2 + \phi'\Lambda' - \Lambda'/2) = -\frac{\omega}{2\psi}\psi'^2 e^{-2\Lambda} - 1/2U \tag{4.5}$$

$$e^{-2\Lambda}\psi'^2(\omega\psi^{-1})_\psi + 2e^{-2\Lambda}\omega\psi^{-1}\left[(2r^{-1} - \Lambda' + \phi')\psi' + \psi''\right] - U_\psi = 0 \tag{4.6}$$

A prime stands for a derivative with respect to r and a ψ indice, a derivative with respect to the scalar field. By subtracting equations (4.3-4.4) and by summing (4.4-4.5), it comes:

$$r(\Lambda' - \phi') - 1 + e^{2\Lambda}\left[1 - 1/2r^2(U+\rho)\right] = 0 \tag{4.7}$$

$$r^2\Lambda'(2\phi' - 1) + 2r^2(\phi'' + \phi'^2) + 4r\phi' + 2 - 2e^{2\Lambda}(1 - r^2U) = 0 \tag{4.8}$$

Then, we derive U and ρ as some functions of Λ and ϕ:

$$\rho = 1/2e^{-2\Lambda}r^{-2}\left[-2 + 2e^{2\Lambda} + r(4 - r + 2r\phi')\Lambda' + 2r^2(\phi'^2 + \phi'')\right] \tag{4.9}$$

$$U = -1/2e^{-2\Lambda}r^{-2}\left[2 - 2e^{2\Lambda} + r\phi'(4 + 2r\phi') + r^2\Lambda'(2\phi' - 1) + 2r^2\phi''\right] \tag{4.10}$$

We introduce these expressions in (4.5) to get ω as a function of Λ, ϕ and ψ. Then putting the above forms of U, ρ and ω in Klein-Gordon equation yields:

$$\Lambda'\left[2(r-2) + r(r-12)\phi' - 2r^2\phi'^2\right] - 2\phi'\left[e^{2\Lambda} - 3 + r^2(\phi'^2 + \phi'')\right] = 0 \tag{4.11}$$

which is scalar field independent. Since we are interested by flat rotation curves, we assume that rotation velocity tends to a constant for large r. However, rotation curves as seen by an observer at infinity for a spherical symmetry, are given by $V_{rot} = \sqrt{rg_{tt,r}/(2g_{tt})}$, as shown in [146] where a newtonian interpretation of this last expression is given. It implies that $\phi' \to V_{rot}^2 r^{-1}$ and then $e^{2\phi} \to r^{2V_{rot}^2}$. To simplify our results below, we define the following constants:

$$\begin{aligned}
c_1 &= 2(1+V_{rot}^2-2V_{rot}^6)\\
c_2 &= V_{rot}^4-1\\
c_3 &= -2(2+6V_{rot}^2+V_{rot}^4)\\
c_4 &= 2+V_{rot}^2\\
c_5 &= -2V_{rot}^2(-3-V_{rot}^2+V_{rot}^4)(2+6V_{rot}^2+V_{rot}^4)^{-1}\\
c_6 &= -3-V_{rot}^2+V_{rot}^4\\
c_7 &= -4-12V_{rot}^2-2V_{rot}^4\\
c_8 &= -2-4V_{rot}^2-4V_{rot}^4
\end{aligned}$$

Introducing ϕ' in (4.11) and integrating, we find for Λ:

$$e^{2\Lambda} = c_6\left[\Lambda_0\left(\frac{c_4 r + c_7}{r}\right)^{c_5} - 1\right]^{-1} \tag{4.12}$$

Λ_0 is a positive integration constant. This last expression is only physically meaning for large r where it tends to the constant $c_6\left[\Lambda_0 c_4^{c_5}-1\right]^{-1}$ as $1/r$. This constant must be positive otherwise e^Λ is not defined for large r and moreover, numerical integrations seem to show that Λ diverges for a finite value of this coordinate. Then, from (4.9), (4.10) and (4.12), we calculate that asymptotically:

$$\rho = 2\left[-1+\Lambda_0\frac{(c_1+c_2 r)(c_3 r^{-1}+c_4)^{c_5}}{c_3+c_4 r}\right](c_6 r^2)^{-1} \tag{4.13}$$

$$U = 2\{c_2 - \frac{(2c_4-3)\Lambda_0\left[c_8+r(c_4-1)\right](c_4+c_3 r^{-1})^{c_5}}{c_3+c_4 r}\}(c_6 r^2)^{-1} \tag{4.14}$$

For large r, ρ and U vanish as r^{-2}. This asymptotical behaviour for the perfect fluid energy density is the same as the nonsingular isothermal profile, one of the most frequent halos. The metric describing the galaxies outer radii where the rotation curves flatten is thus the same as in [146] whatever ω. Considering a perturbation $\delta(r)$ of V_{rot} does not modify these results as long as $r\delta' \to 0$.
From the form of the metric and since we have left ω undetermined, we can get for large r a relation between ω and U such that the rotation curves be flattened. By summing (4.3) and (4.4) and taking into account asymptotical behaviours for ρ and U, we find the following three limits:

$$\omega\psi'^2\psi^{-1} \to 4\ell^{-2}r^{-2} \tag{4.15}$$
$$U \to U_1 r^{-2} \tag{4.16}$$
$$U' = U_\psi\psi' \to -2U_1 r^{-3} \tag{4.17}$$

$\ell^{-2} = \left[c_4 - 3 + \frac{c_6}{c^4(\Lambda_0^{-1}c_4^{-c_5}-1)}\right]$ and U_1 are some constants. We use (4.16) and (4.17) for respectively introduce U and replace ψ' in (4.15). Then, considering g and k two functions of r and rewriting (4.15) and (4.17) as respectively $\omega\psi'^2\psi^{-1} \to g(r)$ and $U_\psi\psi' \to k(r)$, we have

$$\frac{\omega k^2}{\psi U_\psi^2} \to gr^4 r^{-4}$$

Using (4.16) to replace r^4 and introduce U, it comes

$$\frac{\omega U^2}{\psi U_\psi^2} \to U_1^2\frac{g(r)}{k(r)^2}r^{-4}$$

Since here $g(r) = 4\ell^{-2}r^{-2}$ and $k(r) = -2U_1 r^{-3}$, we find:

$$\frac{\psi U_\psi^2}{\omega U^2} \to \ell^2 \neq 0 \tag{4.18}$$

For a given form of $U(\psi)$, (4.16) defines a unique form for $r(\psi)$. $U(\psi)$ and $r(\psi)$ may then be introduced in (4.17), defining a unique form for $\psi'(\psi)$. Then, $r(\psi)$ and $\psi'(\psi)$ may be introduced in (4.15), defining a unique form for $\omega(\psi)$. Consequently, for a given $U(\psi)$, (4.15-4.16) defined a unique $\omega(\psi)$. The same remark applies to (4.18). Consequently, for a given U, the system (4.15-4.17) or (4.18) uniquely define ω and thus the class of scalar tensor-theories responsible for the rotation curves flattening for given potential or Brans-Dicke coupling function. However, only (4.15-4.17) uniquely define $\psi(r)$.
We have shown above that ρ asymptotically behaves as r^{-2}. Examining the equation (4.3), we note that the scalar field energy density ρ_ϕ shall be written as $\frac{\omega}{2\psi}\psi'^2 e^{-2\Lambda}+1/2U$. Knowing the asymptotical limits of each of these terms, we deduce that for large r, $\rho\propto\rho_\phi$: the scalar field is quintessent.
The limit (4.18) is doubly important. Firstly, in [105, 127, 116] it has been shown that a necessary condition for isotropisation of Bianchi models was $\frac{\psi U_\psi^2}{\omega U^2}\rightarrow\ell^2$, ℓ being a constant in a close interval depending on the presence of curvature and perfect fluid. Consequently, galactic scalar field properties for large r could match a cosmological scalar field present in the entire Universe which would allow for its isotropisation. Secondly, specifying one of the unknown functions ω or U, (4.18) allow determining in a unique way the other one: this limit gives a necessary and sufficient relation between these two functions such that the galactic rotation curves for outer radii be flattened. It thus generalises the work of [146] for which ω was a known function of the scalar field leading to an exponential potential.
In the following section, we examine the stability of these results with respect to the presence of a second scalar field.

4.2.2 With 2 scalar fields

When two massive scalar fields are present, the action may take the following form:

$$S=\int(R-\frac{\omega_1}{\psi_1}\psi_{1,\mu}\psi_1^{;\mu}-\frac{\omega_2}{\psi_2}\psi_{2,\mu}\psi_2^{;\mu}-U+\frac{16\pi}{c^4}L_m)\sqrt{-g}d^4x \tag{4.19}$$

The ψ_i are two scalar fields such that $\omega_1=\omega_1(\psi_1)$, $\omega_2=\omega_2(\psi_2)$ and $U=U(\psi_1,\psi_2)$. This form of the action is not the most general one but allows testing the results of the previous section. The field equations are:

$$r^{-2}\left[r(1-e^{-2\Lambda})\right]'=\frac{\omega_1}{2\psi_1}\psi_1'^2e^{-2\Lambda}+\frac{\omega_2}{2\psi_2}\psi_2'^2e^{-2\Lambda}+1/2U+\rho \tag{4.20}$$

$$-r^{-2}(1-e^{-2\Lambda})+2r^{-1}\phi'e^{-2\Lambda}=\frac{\omega_1}{2\psi_1}\psi_1'^2e^{-2\Lambda}+\frac{\omega_2}{2\psi_2}\psi_2'^2e^{-2\Lambda}-1/2U \tag{4.21}$$

$$e^{-2\Lambda}(\phi''+\phi'^2+\phi'\Lambda'-\Lambda'/2)=-\frac{\omega_1}{2\psi_1}\psi_1'^2e^{-2\Lambda}-\frac{\omega_2}{2\psi_2}\psi_2'^2e^{-2\Lambda}-1/2U \tag{4.22}$$

$$e^{-2\Lambda}\psi_1'^2(\omega_1\psi_1^{-1})_{\psi_1}+2e^{-2\Lambda}\omega_1\psi_1^{-1}\left[(2r^{-1}-\Lambda'+\phi')\psi_1'+\psi_1''\right]-U_{\psi_1}=0 \tag{4.23}$$

$$e^{-2\Lambda}\psi_2'^2(\omega_2\psi_2^{-1})_{\psi_2}+2e^{-2\Lambda}\omega_2\psi_2^{-1}\left[(2r^{-1}-\Lambda'+\phi')\psi_2'+\psi_2''\right]-U_{\psi_2}=0 \tag{4.24}$$

Making the same calculus as in section 4.2.1, we get an equation similar to (4.11), i.e. independent on the scalar fields:

$$\begin{aligned}&4-4e^{2\Lambda}+4r^3\phi'^3+2r^3\Lambda'^2(2\phi'-1)+r^3\Lambda''-2r\phi'(4-2e^{2\Lambda}+r^2\Lambda'')-\\&\quad 4r^2\phi''+2r\Lambda'[6-2r-(r-16)r\phi'+4r^2\phi'^2+r^2\phi'']-2r^3\phi'''=0\end{aligned} \tag{4.25}$$

Equations for ρ and U are the same as (4.9) and (4.10). Equation (4.25) does not depend on U, ω_1, ω_2 or the scalar fields forms. Moreover, we always have $\phi'\rightarrow V_{rot}^2r^{-1}$ which asymptotically characterises a flat rotation curve. The solution for Λ issued from equation (4.25) is thus independent on the scalar fields and the unknown functions ω_i and U. It will be always the same, whatever U, ω_1, ω_2 and ψ_i. In particular, if we consider the special case where one of the scalar fields is negligible, one have to recover the same asymptotical form for Λ as when only one scalar field is present. Hence, the asymptotical solution for equation (4.25) should be the same as for (4.11): when 2 scalar fields are considered, Λ tends to a constant as r^{-1} and Λ' vanishes as r^{-2}. This is in agreement with [147] and implies that U and ρ also vanish as r^{-2}. These results are the same if we consider a perturbation δ for the rotation velocity as long as $\delta' r$ and $\delta'' r^2$ are asymptotically vanishing.

Anew, we find the following limits allowing to determine if a relation exists between ω_1, ω_2 and U when the rotation curves flatten:

$$\omega_1\psi_1'^2\psi_1^{-1}+\omega_2\psi_2'^2\psi_2^{-1} \rightarrow 2\ell^{-2}r^{-2} \quad (4.26)$$
$$U \rightarrow U_1 r^{-2} \quad (4.27)$$
$$U'=U_{\psi_1}\psi_1'+U_{\psi_2}\psi_2' \rightarrow -2U_1 r^{-3} \quad (4.28)$$

Let us put that $\omega_i\psi_i'^2\psi_i^{-1}\rightarrow g_i$ and $U_{\psi_i}\psi_i'\rightarrow k_i$. Then, $g_1+g_2\rightarrow r^{-2}$, $k_1+k_2\rightarrow r^{-3}$ and we have

$$\frac{\omega_i U^2}{\psi_i U_{\psi_i}^2}\rightarrow U_1^2\frac{g_i}{k_i^2}r^{-4}$$

It implies that only one of the g_i(or k_i) have to tend to r^{-2} (respectively r^{-3}), the second one varying as or slower than this last limit. Moreover, equations (4.23-4.24) may be written as:

$$(\frac{\omega_i}{\psi_i}\psi_i'^2e^{-2\Lambda})'+2\frac{\omega_i}{\psi_i}\psi_i'^2e^{-2\Lambda}(\frac{2}{r}+\phi')-U_{\psi_i}\phi_i'=0$$

Hence, since $\Lambda\rightarrow const$ and $\phi'\rightarrow r^{-1}$, if $g_i<r^{-2}$, k_i have to vary slower than r^{-3}. We thus distinguish 2 possible behaviours for g_i and k_i:

- case 1: $g_i\rightarrow r^{-2}$ and $k_i\rightarrow r^{-3}$
  As previously, it comes that $\frac{\psi_1 U_{\psi_1}^2}{\omega_1 U^2}$ and $\frac{\psi_2 U_{\psi_2}^2}{\omega_2 U^2}$ tend to some constants.
- case 2: $g_1\rightarrow r^{-2}$, $g_2<<r^{-2}$, $k_1\rightarrow r^{-3}$ et $k_2<<r^{-3}$
  Consequently, the dynamical effects of ψ_2 are asymptotically negligible in the field equations and the metric functions dynamics does not depend on it. We find that $\frac{\omega_1 U^2}{\psi_1 U_{\psi_1}^2}$ tends to a non vanishing constant and $\frac{\omega_2 U^2}{\psi_2 U_{\psi_2}^2}$ diverges or vanishes.

We conclude that the scalar fields ψ_i which are not asymptotically negligible are such that $\frac{\psi_i U_{\psi_i}^2}{\omega_i U^2}$ tend to some non vanishing constants. Hence, the results of the previous section are not modified by the introduction of a second scalar field. Anew, we note that the scalar fields energy density which shall be $\frac{\omega_1}{2\psi_1}\psi_1'^2e^{-2\Lambda}+\frac{\omega_2}{2\psi_2}\psi_2'^2e^{-2\Lambda}+1/2U$ asymptotically tends to r^{-2} and behaves as the one of the dust fluid. It means that the two (or the dominant) scalar fields are (respectively is) quintessent.

4.3 Discussion

In this work, we have looked for the characteristics of the metric functions and scalar field such that galactic rotation curves flatten for outer radii. For the metric we have got the following result:

Let us consider a minimally coupled and massive scalar field defined by a potential U and a Brans-Dicke coupling function ω with a spherically static metric. When, at outer radii, the galactic rotation curves flatten, the metric is asymptotically defined by $ds^2=-r^{2V_{rot}^2}dt^2+\Lambda_1 dr^2+r^2(d\theta^2+sin^2\theta d\phi^2)$, Λ_1 being a constant, whatever the form of the potential or Brans-Dicke coupling function.

This result is stable related to a small perturbation δ of the rotation velocity such that $r\delta'\rightarrow 0$ or if we consider a second scalar field ϕ as defined by the Lagrangian (4.19). In this last case the perturbation must be such that $\delta' r$ and $\delta'' r^2$ vanish for large r. It is in accordance with the asymptotical form of the metric found in [147] where an exponential potential was found to explain the flattening of the rotation curves. Here, this property is generalized to any functions ω and potential U with the following characteristics:

Let us consider a minimally coupled and massive scalar field defined by a potential U and a Brans-Dicke coupling function ω with a spherically static metric. When, at outer radii, the galactic rotation curves flatten, the energy densities of the perfect fluid and scalar field vanish as r^{-2}: the scalar field is asymptotically quintessent. Moreover, ω and U are asymptotically related by the relation $\frac{\psi U_\psi^2}{\omega U^2}\rightarrow\ell^2$, ℓ^2 being a constant, and U vanishes as r^{-2}.

Hence the energy density of the scalar field shows that we are using an isothermal profile which decays in r^{-2} and fit the flat galactic rotation curves quite well and not a Navarro-Frenk-White [151] profile where the density goes like r^{-3} in the asymptotic region and whose corresponding metric and rotation velocity has been recently determined in [152]. When 2 scalar fields are present, again their energy density behaves as the one of the perfect fluid. Consequently, at least one of them is quintessent. The presence of quintessent scalar fields in spiral galaxies have been examined in [153] where the agreement with the observed rotation curves is shown. Moreover, the scalar fields which are not negligible are such that $\frac{\psi_i U^2_{\psi_i}}{\omega_i U^2}$ tends to a constant, leaving the above last result unchanged for these scalar fields.

Let us make some remarks on the quantity $\frac{\psi U^2_{\psi}}{\omega U^2}$. Firstly, *any necessary condition expressed with the unknown functions*[2] *of a scalar tensor theory such that the metric converge toward a determined form must be invariant with respect to a scalar field transformation*. Indeed, considering $F(U,\omega)$, a necessary condition such that ds^2 always tends to a determined form, since a transformation $\psi = T(\Psi)$ of the scalar field keeps the metric, it must be the same for the necessary condition $F(U,\omega)$. One can easily check it is the case for $\frac{\psi U^2_{\psi}}{\omega U^2}$. Particularly there is a scalar field transformation which allows to rewrite the metric under the form

$$S = \int (R - \Psi_{,\mu}\Psi^{,\mu} - U + \frac{16\pi}{c^4}L_m)\sqrt{-g}d^4x$$

corresponding to the one of [146] and leading to their results. However, for most of ω functions, this transformation is not defined or analyticaly workable and thus the results of [146] can not be arbitrarily applied to any forms of couple (ω,U) whereas it could be important to keep both ω and U as depending on the scalar field. As instance, if the potential vanishes, thus mimicing a vanishing cosmological constant, compatibility of the theory with PPN parameters requires that $\omega \to \infty$ and $\omega_{\phi}\omega^{-3} \to 0$ [56, 57], which does not fit with a constant ω got after field redefinition. Indeed, it recovers the general problem of finding a metric whose dynamics is agreed with the observations and whose potential is in accordance with, as instance, particle physics predictions for the form of the potential: for this, it is necessary to keep the freedom of choosing a form for $\omega(\phi)$. Keeping ω as an undetermined function of the scalar field thus allows finding the set of all scalar tensor theories able to produce flat rotation curves for *any* forms of ω and U, even when the above scalar field redefinition can not be anatically performed.

The part of the results concerning the convergence of $\frac{\psi U^2_{\psi}}{\omega U^2}$ to a constant seems strangely correlated to isotropisation of homogeneous models which also needs this condition [105, 127, 116]. It shows that the properties of a galactic scalar field allowing the flattening of rotation curves could match those of a cosmological scalar field favouring Universe isotropisation.

Starting from the form of the potential, the property $\frac{\psi U^2_{\psi}}{\omega U^2} \to \ell^2$ allows recovering the Brans-Dicke coupling function such that the rotation curves could flatten and vice-versa. Let us examine some of the most studied potentials. Hence, if we consider an exponential potential $U = e^{k\psi}$, we find that $\omega = k^2\ell^{-2}\psi$, in accordance with the results of [146] as a particular case of the class of theories we have found, and $\psi \propto \ln r$. If we take $U = \psi^k$, the Brans-Dicke coupling function should be $\omega = k^2\ell^{-2}\psi^{-1}$ and $\psi \propto r^{-2k^{-1}}$. Thus, we see that considering two unknown functions ω and U instead of a single one leads to an important generalisation of [146]. Indeed, assuming asymptotically flat rotation curves fix the behaviour of one of the metric function, i.e. ϕ. Consequently, in [146], since ω is chosen, the potential is uniquely determined whereas in this paper we can find it depends on ω, thus explaining the asymptotical relation between ω and U. It follows that an exponential potential is not the only one allowing to get flat rotation curves but a whole class of scalar tensor theories such that $\frac{\psi U^2_{\psi}}{\omega U^2} \to \ell^2$ leads to this property. We remark that a cosmological constant can not explain why the rotation curves flatten since the potential must evolve as r^{-2}. This is in agreement with the theories which try to solve the cosmological constant problem by considering it as a variable function rather than a true constant.

The interpretation of scalar fields properties found in this work may be done at cosmological or galactic scales. Phenomena giving birth to scalar fields probably have a cosmological nature and are related to particle physics theories. To our knowledge it does not exist any way to generate them by galactic process. Their properties based on rotation curves observations are coherent with their usual cosmological picture i.e. their quintessent nature and their role in the Universe isotropisation. However, if we consider an asymptotically increasing rotation curve, it could mean that ϕ' increased faster than r^{-1}. It would come from equations (4.10) and (4.9) that the quantity U/ρ could diverge. If such rotation curves were observed (it is

2. i.e. ω and U for the present case.

the case but some doubts subsist on the fact that they could asymptotically flatten [149]), it would mean that scalar fields, at least at galactic scale, would not be quintessent. Then the quintessence properties would only be valid for some types of galaxies. In [148], decreasing, increasing or flattening rotation curves are studied. Observations seem to show that the first ones appear for bright galaxies and the second ones for faint galaxies. A possible interpretation would be that flat rotation curves would be the outcome of an equivalent mixing between luminous and dark matter, the decreasing or increasing rotation curves resulting of a respectively luminous matter or dark matter domination. This flat rotation curves interpretation is coherent with the scalar field quintessence property found in this paper.

To conclude, this work generalises those of [146] and [147]. The metric got in these papers and the quintessent nature of the scalar fields have been generalised for any form of ω. A relation between U and ω has been found and allows getting easily one of these quantities from the other. It selects the class of scalar tensor theories which could be in agreement with flat rotation curves and shows that an exponential potential is not the only one able to produce such a dynamics. Moreover, we have remarked that this class is also in agreement with Universe isotropisation. The stability of these results with respect to a small perturbation of rotation velocity or the presence of a second scalar field have been tested. A next step would be to consider a non minimally coupled scalar tensor theory, i.e. with a variable gravitational constant or/and a magnetic field that could play a fundamental role at a galactic scale.

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Cinquième partie

Conclusion et perspectives

Dans cette thèse, nous avons considéré les propriétés des modèles cosmologiques homogènes en théories tenseur-scalaires et nous avons cherché à contraindre ces théories en étudiant ces modèles. Nous avons procédé en deux étapes.

Dans une première étape, nous avons recherché quelles caractéristiques (comportements asymptotiques [154, 155], singularités[156, 157], symétries[158], etc) un Univers homogène devait posséder et quelles méthodes utiliser (solutions exactes[159, 160], études asymptotiques[161], formalisme Hamiltonien[42], etc) afin de définir une théorie tenseur-scalaire en accord avec ces caractéristiques. Ces méthodes se basent essentiellement sur:

– la recherche de solutions exactes
  Ceci ne s'applique qu'à un nombre très restreint de théories mais a été, de notre point de vue, un excellent moyen de découvrir la grande variété des théories tenseur-scalaires, des modèles de Bianchi et les sources de leur complexité. De plus, la recherche de solutions exactes est présente tout au long de notre travail, nous permettant de vérifier et de valider les résultats issus d'études plus générales. Elles représentent donc en cela un élément clef.
– la considération d'hypothèses purement théoriques
  Considérer qu'un Univers homogène est dépourvu de singularité ou respecte une symétrie de Noether permet de fortement contraindre une théorie tenseur-scalaire. Dans le premier cas cependant nous n'avons pas pu étudier de théories possédant des champs scalaires massifs. Dans le second cas, nos résultats ont été obtenus pour les modèles isotropes FLRW. Nous avons tenté de les étendre aux modèles anisotropes mais sans réel succès. Il semblerait que la méthode de calculs des symétries de Noether doive être adaptée d'une manière ou d'une autre.
– des considérations d'ordre dynamique
  L'expansion de l'Univers, l'accélération de cette expansion ou son isotropie sont autant de phases dynamiques que l'Univers traverse ou a traversé de manière certaine et qui peuvent nous servir à contraindre les théories tenseur-scalaires afin qu'elles les reproduisent.

Pour cette première étape, nous concluons qu'un grand nombre de modèles cosmologiques anisotropes et de théories tenseur-scalaires peuvent être contraints en étudiant leurs équations de champs issues du formalisme Hamiltonien[78] à l'aide des méthodes d'analyse des systèmes dynamiques[25] et en recherchant les processus menant à l'isotropisation[105].

La deuxième étape a alors consisté à appliquer cette méthode à l'ensemble des modèles de Bianchi de la classe A[109, 127, 129] et à des théories tenseur-scalaires possédant jusqu'à trois fonctions indéterminées du champ scalaire[116, 162](fonction de gravitation, potentiel, fonction de Brans-Dicke, etc).
Du point de vue de l'analyse des systèmes dynamiques, nous avons détecté trois familles de points d'équilibre, correspondant à trois manières différentes pour l'Univers de s'isotropiser et les avons appelées classe 1, 2 et 3. Nous nous sommes intéressés à la classe 1 et avons obtenu des résultats consistant en:

1. La localisation des points d'équilibre isotropes stables.
2. Les conditions nécessaires à leur existence et contraignant les théories tenseur-scalaires.
3. Les comportements asymptotiques du champ scalaire, des fonctions métriques et du potentiel.

Ceux de ces résultats décrivant où liés à des comportements asymptotiques ont été calculés sous l'hypothèse que l'Univers tend suffisamment vite vers son état d'équilibre. Mathématiquement parlant cela signifie qu'à l'approche de l'isotropie, les diverses variables figurant dans les équations de champs tendent suffisamment vite vers leurs valeurs à l'équilibre afin que l'on puisse négliger leurs variations dans les calculs. Nous avons montré comment cette hypothèse pouvait être levée en ce qui concerne la fonction ℓ. En revanche pour les autres variables, une étude des perturbations à l'approche de l'équilibre s'avère nécessaire et des progrès devront être faits dans ce sens pour compléter l'étude des processus d'isotropisation de classe 1.

Au final, l'état dans lequel se trouve l'Univers lorsqu'il atteint l'isotropie présente des caractéristiques intéressantes. En particulier pour les champs scalaires minimalement couplés, cet état isotrope peut être résumé de la manière suivante:

– L'univers est en expansion tel que les fonctions métriques tendent vers des puissance ou des exponentielles du temps propre et le potentiel respectivement disparaît comme t^{-2} ou tend vers une constante.

- La présence de courbure favorise une accélération tardive et la quintessence.
- L'univers est asymptotiquement plat.
- Il existe un lien entre les théories tenseur-scalaires menant à l'isotropisation et celles permettant un aplatissement des courbes de rotation des galaxies spirales.

Lorsque le champ scalaire est non minimalement couplé, nous avons pu contraindre les théories tenseur-scalaires de telle façon qu'elles soient compatibles avec l'isotropisation et déterminer les comportements asymptotiques des fonctions dans le référentiel d'Einstein mais il est impossible d'obtenir ces comportements sans quadrature dans le référentiel de Brans-Dicke.

L'ensemble de ces résultats est illustré par de nombreuses applications analytiques et numériques. Ces dernières ont été réalisées à l'aide de méthodes de Runge-Kutta que nous avons implémentées en Java. Java est un langage peu utilisé en sciences où on lui préfère fortran. Il possède pourtant de nombreuses qualités, la première étant d'être totalement gratuit. D'un point de vue technique, c'est un langage objet et ceci nous a permis de séparer les méthodes d'intégration des systèmes à intégrer en classes distinctes. Ceci fait, pour intégrer un système d'équation avec une méthode de Runge-Kutta, il n'y a plus qu'à écrire les équations sous forme d'une nouvelle classe et il n'est plus besoin de réimplémenter la méthode numérique. Evidement, la même chose est possible avec Fortran mais la structure du développement est alors beaucoup moins claire et ce langage n'est pas portable. En effet, une fois l'application java compilée, elle fonctionne sur n'importe quel environement (windows, linux, mac, etc), diminuant ainsi les coups de développement. Enfin, il est également vrai que des intégrations numériques peuvent être faites avec des logiciels comme Mathématica mais elles manquent de souplesses car l'on ne dispose pas des codes sources de ces applications et l'on est donc limité par leur langage.

Ce qui reste à accomplir est encore immense mais nous espérons avoir défini un cadre de travail opérationnel capable de guider de futures recherches sur le sujet de l'isotropisation des cosmologies homogènes mais anisotropes en théories tenseur-scalaires. Entre autres perspectives, il serait important de prendre en compte les cas de potentiels ou de fonctions de Brans-Dicke négatifs. Ceci correspondrait alors à un non respect de la condition d'énergie faible, hypothèse qui revient constamment dans la littérature mais qui manque encore de motivations physiques. Plus important, il serait utile de généraliser nos résultats aux modèles dont la convergence vers l'état isotrope n'est pas suffisamment rapide pour négliger les termes du second ordre pour ℓ (hypothèse de variabilité) ou pour les variables dont nous nous sommes servi pour réécrire les équations Hamiltoniennes. Enfin, une dernière possibilité importante consisterait en l'étude des classes d'isotropisation de type 2 et 3 qui n'ont été abordées que numériquement à travers des applications. Un axe de recherche possible serait donc d'obtenir des résultats analytiques. En particulier la classe 3, s'est révélée être le moyen d'isotropisation privilégier de certaines théories tenseur-scalaires possédant un champ scalaire complexe comme le montre la discussion de la section 1.3 de la partie IV.

Sixième partie

Appendice

Cet appendice reproduit les articles publiés (3), acceptés pour publication (1) ou soumis (2) dans Classical and Quantum Gravitation concernant l'isotropisation des modèles de Bianchi. Leur contenu est résumé et parfois étendu dans la partie IV de cette thèse.

Chapitre 1

Isotropisation of Generalized Scalar-Tensor theory plus a massive scalar field in the Bianchi type I model

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Abstract

In this paper we study the isotropisation of a Generalized Scalar-Tensor theory with a massive scalar field. We find it depends on a condition on the Brans-Dicke coupling function and the potential and show that asymptotically the metric functions always tend toward a power or exponential law of the proper time. These results generalise and unify these of De Sitter in the case of a cosmological constant and of Cooley and Kitada in the case of an exponential potential.

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1.1 Introduction

In this paper we wish to study the conditions for isotropisation of the Generalized Scalar-Tensor theory plus a massive scalar field in the Bianchi type I model. They are many reasons to be interesting by this model

First of all General Relativity is a good description for the weak gravitational fields (solar system tests) as for strong ones (binary pulsar) although deviating are expected for early times or in extreme cases such as black hole. Then, it is interesting to consider a Lagrangian whose "geometric" part looks like General Relativity. Moreover particle physics progress and the idea of inflation in the eighties show that scalar fields could be essential components of a gravitational theory. In this work we will consider a massive one. It could be justified by observations of type IA supernovae [9, 10] which seem to demonstrate that the dynamical behaviour of our Universe is accelerated. Most of time this is interpreted as the presence of a cosmological constant in the field equations although other explanations can be advanced such as this of a non-perfect fluid with quintessential matter [163]. The Boomerang experiment [37] indicates that it could represent the dominant energy of our Universe. However, the present value of this constant is in disagreement with this predicted by particle physics at early times. One way to solve this problem is to consider a varying potential U, i.e. a massive scalar field. We will also describe the coupling of the scalar field ϕ with the metric functions by a coupling function $\omega(\phi)$ generalising the Brans-Dicke coupling constant [7]. This type of coupling is issued from particle physics theories whose Lagrangian at low energy could take the form of a scalar tensor theory.

Geometrically, our present Universe seems well described by the isotropic and homogeneous cosmological models, i.e. the FLRW models. However the observations, as instance from Boomerang [37], show slight

anisotropies in the cosmological microwave background, which could take origin at early times. Moreover, if the Universe had always been perfectly isotropic and homogeneous, it would be difficult to explain the large-scale structures we observe. Hence, it is interesting to consider an anisotropic Universe described by the Bianchi models. There are 9 ones but the most studied are the Bianchi type I, V, VII_0, VII_h and IX which are able to isotropise toward an FLRW model [108]. We will consider the Bianchi type I model which can tend toward a flat FLRW one and is a good candidate from the inflation theory point of view. Of course the Bianchi models are not a definitive geometrical description of the Universe which should probably be inhomogeneous. But such models allow studying the necessary conditions for its isotropisation.
Our goal will be to look for the necessary conditions depending on the potential and the Brans-Dicke coupling function for Universe isotropisation at late times. We will then derive the asymptotical dynamical behaviour of the metric functions and the condition for the presence of inflation.
Technically, we will use the ADM Hamiltonian formalism [78, 79] allowing to write the field equations as a first order system and then the dynamical systems theory as described in [25] and suggested in [42] to study them. We have not found any paper in the literature where these two methods are applied to equations system with 2 arbitrary functions. Due to this indeterminacy all the equilibrium points of the system can not be studied. However this problem can be overcomen for the subset of the phase space where lie the isotropic states of the Universe and which is of interest for us.
This paper is organised as follow. In the second section we calculate the Hamiltonian field equations of the Generalized Scalar-Tensor theory plus a massive scalar field and rewrite them with new normalised variables. In the third section, we study the subset of the phase space corresponding to isotropy. We discuss about physical meaning of the mathematical results thus obtained in the fourth section.

1.2 Field equations

The Lagrangian of the Generalized Scalar-Tensor theory plus a massive scalar field is written:

$$S = (16\pi)^{-1} \int \left[R - (3/2 + \omega(\phi))\phi^{,\mu}\phi_{,\mu}\phi^2 - U(\phi)\right]\sqrt{-g}d^4x \tag{1.1}$$

with ϕ the scalar field, ω the coupling function between the scalar field and the metric, U the potential. We will use the following form of the metric:

$$ds^2 = -(N^2 - N_i N^i)d\Omega^2 + 2N_i d\Omega\omega^i + R_0^2 g_{ij}\omega^i\omega^j \tag{1.2}$$

the ω^i being the 1-forms defining the Bianchi type I homogeneous space. The g_{ij} are the metric functions, N and N_i are respectively the lapse and shifts functions. Using the methods described in [78] and [77], we find that the action can be rewritten in the following way:

$$S = (16\pi)^{-1} \int (\Pi^{ij}\frac{\partial g_{ij}}{\partial t} + \Pi^\phi \frac{\partial \phi}{\partial t} - NC^0 - N_i C^i)d^4x \tag{1.3}$$

The Π^{ij} and Π^ϕ are respectively the conjugate momentum of the metric functions and scalar field, the N and N_i play the role of Lagrange multipliers. The quantities C_0 and C_i are respectively the super-Hamiltonian and supermomentum defined by:

$$C^0 = -\sqrt{^{(3)}g}^{(3)}R - \frac{1}{\sqrt{^{(3)}g}}(\frac{1}{2}(\Pi_k^k)^2 - \Pi^{ij}\Pi_{ij}) + \frac{1}{\sqrt{^{(3)}g}}\frac{\Pi_\phi^2\phi^2}{6+4\omega} + \sqrt{^{(3)}g}U(\phi) \tag{1.4}$$

$$C^i = \Pi^{ij}_{|j} \tag{1.5}$$

where the "$^{(3)}$" hold for the quantities calculated on the 3-space and the "|" for the covariant derivative in the 3-space. When we vary the action with respect to the Lagrange multipliers we find the constraints $C^0 = C^i = 0$.
We rewrite the metric functions as $g_{ij} = e^{-2\Omega + 2\beta_{ij}}$ and we use the Misner parameterisation [79]:

$$p_k^i = 2\pi\Pi_k^i - \frac{2}{3}\pi\delta_k^i\Pi_l^l \tag{1.6}$$

$$6p_{ij} = diag(p_+ + \sqrt{3}p_-, p_+ - \sqrt{3}p_-, -2p_+) \tag{1.7}$$

$$\beta_{ij} = diag(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+) \tag{1.8}$$

Then, the action (1.3) is written as:

$$S = \int p_+ d\beta_+ + p_- d\beta_- + p_\phi d\phi - H d\Omega \tag{1.9}$$

with $p_\phi = \pi\Pi_\phi$ and $H = 2\pi\Pi^k_k$. Finally, from the constraint $C^0 = 0$, we get the expression for the ADM Hamiltonian:

$$H^2 = p_+^2 + p_-^2 + 12\frac{p_\phi^2\phi^2}{3+2\omega} + 24\pi^2 R_0^6 e^{-6\Omega}U \tag{1.10}$$

from which we derive the Hamiltonian equations:

$$\dot{\beta}_\pm = \frac{\partial H}{\partial p_\pm} = \frac{p_\pm}{H} \tag{1.11}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{12\phi^2 p_\phi}{(3+2\omega)H} \tag{1.12}$$

$$\dot{p}_\pm = -\frac{\partial H}{\partial \beta_\pm} = 0 \tag{1.13}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = -12\frac{\phi p_\phi^2}{(3+2\omega)H} + 12\frac{\omega_\phi \phi^2 p_\phi^2}{(3+2\omega)^2 H} - 12\pi^2 R_0^6 \frac{e^{-6\Omega}U_\phi}{H} \tag{1.14}$$

$$\dot{H} = \frac{dH}{d\Omega} = \frac{\partial H}{\partial \Omega} = -72\pi^2 R_0^6 \frac{e^{-6\Omega}U}{H} \tag{1.15}$$

A dot means a derivative with respect to Ω. We will choose $N^i = 0$ and we calculate N by writing that $\partial\sqrt{g}/\partial\Omega = -1/2\Pi^k_k N$ [78]. Then, it comes:

$$N = \frac{12\pi R_0^3 e^{-3\Omega}}{H} \tag{1.16}$$

The relation between the Ω and t times is then $dt = -N d\Omega$. We want to rewrite some of these equations under the form of an autonomous first order system with normalised variables [25]. For this we will only consider the set of equations (1.12), (1.14) and (1.15), the equations (1.13) showing that the conjugate momentums of the variables describing the anisotropy are some constants. The constraint (1.10) suggests the following set of normalised variables:

$$x = H^{-1} \tag{1.17}$$

$$y = \sqrt{e^{-6\Omega}U}H^{-1} \tag{1.18}$$

$$z = p_\phi \phi (3+2\omega)^{-1/2} H^{-1} \tag{1.19}$$

The first one depends on H, the second one on H and ϕ and the third one on H, ϕ and p_ϕ. Thus they are independent variables. y and z will be real if the functions U and $3+2\omega$ are positives. Under these conditions, it follows that the potential will favour inflation and the coupling function, when $U = 0$, will be such that the energy density of the scalar field is positive. Rewriting the constraint (1.10) with the new variables, we get:

$$p^2x^2 + R^2y^2 + 12z^2 = 1 \tag{1.20}$$

The positive constants p and R^2 are defined by $p^2 = p_+^2 + p_-^2$ and $R^2 = 24\pi^2 R_0^6$. From this last equation we deduce that the variables x, y and z are bounded and belong to the following intervals:

$$x \in \left[-p^{-1}, p^{-1}\right] \tag{1.21}$$

$$y \in [-R, R] \tag{1.22}$$

$$z \in \left[-1/\sqrt{12}, 1/\sqrt{12}\right] \tag{1.23}$$

The field equations (1.12), (1.14) and (1.15), become (see appendix 1.5):

$$\dot{x} = 3R^2y^2x \tag{1.24}$$

$$\dot{y} = y(6\ell z + 3R^2y^2 - 3) \tag{1.25}$$

$$\dot{z} = y^2R^2(3z - \frac{1}{2}\ell) \tag{1.26}$$

with $\ell = \phi U_\phi U^{-1}(3+2\omega)^{-1/2}$. They can not be expressed only with x, y and z because we do not wish to specify the form of ω and U which are arbitrary functions of the scalar field. However, we do not need to know the exact form of $\ell(x,y,z)$ since we are only interested by the asymptotical isotropisation of the Universe at late times. To reach our goal, it is sufficient to assume two types of asymptotical behaviours for ℓ: ever it tends toward a constant or it diverges. This excludes any asymptotical chaotic behaviour for ℓ and is in accordance with much of the functions ω and U studied in the literature. We will have to check if these behaviours are compatible with the isotropisation of the Universe at late times.

In the next section, we examine the equations (1.24-1.26) from the dynamical systems theory point of view. Firstly, we look for monotonic functions and secondly, we study the presence of equilibrium points.

1.3 Dynamical studies of the fields equations

Monotonic functions

Lets examine the presence of monotonic functions. From the equation (1.24), we deduce that x is a monotonic function: when it is positive (negative), it increases (decreases). Since x has a constant sign, it follows from (1.18) that it is the same for y. Notes also that in the plane $x=0$ with $\ell = cte$, z is a monotonic and increasing function if $z > \ell/6$, decreasing otherwise. Thus there is no periodic or homoclinic orbit and then no chaotic behaviour.
If we look for the signs of the derivatives of the metric functions with respect to Ω depending on the position of a point (x,y,z) in the phase space, we see that the sets of points such that they are constants are splat by planes defined by $x = cte$ since $dg_{ij}/d\Omega = -2e^{-2\Omega+2\beta_{ij}}(1-\dot{\beta}_{ij})$. As instance for g_{11}, it is defined by $x = (p_+ + \sqrt{3}p_-)^{-1}$ and the sign of its derivative above or below this plane depends on the value of the constant $p_+ + \sqrt{3}p_-$. For g_{22}, the plane is $x = (p_+ - \sqrt{3}p_-)^{-1}$ and for g_{33}, $x = p_+^{-1}$. Since x is a monotonic function with constant sign, we deduce that each metric functions can have one and only one extremum. From (1.16) and the relation between Ω and t, we remark it will be the same in the proper time t. This is in agreement with the results of [42]

Study of the isotropic equilibrium states

Now we study the equilibrium points. They are all defined by $(y,z) = (\pm(3-\ell^2)^{1/2}(3R^2)^{-1/2}, \ell/6)$ and they will respect the constraint if $x=0$. We have shown in [42] that the Universe isotropises in the proper time t only when $\Omega \to -\infty$. This value of Ω indicates that they will be sources or sinks but not saddle points. Thus an equilibrium states will represent an isotropic one for the Universe if in the same time Ω diverges negatively. Moreover, since x is a monotonic function of constant sign, we deduce from the relation (1.16) that the proper time is a monotonic and decreasing (increasing) function of Ω when the Hamiltonian is positive (negative). Hence Ω can be considered as a time variable and the equilibrium will take place at late times if $H > 0$. In what follows, we will assume that ℓ asymptotically tends toward a constant or diverges.

First, we assume that ℓ is asymptotically a constant. We can show in that case by integrating (1.25-1.26) that when $y = \pm(3-\ell^2)^{1/2}(3R^2)^{-1/2}$, Ω diverges. It follows that these two equilibrium points are compatible with the isotropisation of the Universe. R^2 being a positive constant, they will be real if $\ell^2 < 3$. We can not calculate their corresponding eigenvalues and thus knowing the signs of these last quantities because we do not now the expressions of the derivatives of ℓ with respect to x, y and z. However, since we consider Ω as a time variable, they will be sinks if $H > 0$ or sources if $H < 0$ since they will respectively correspond to asymptotical late or early times.
To get the behaviour of the metric functions when we approach an isotropic state, we need a differential equation for x when $\ell \to cte$. In this last case, the integration of the field equations (1.24-1.26) gives:

$$z = \left[\ell(1+6R^2z_0) - \sqrt{\ell^2(1+6R^2z_0)+18R^2z_0(R^2y^2-1)}\right](36R^2z_0)^{-1} \tag{1.27}$$

By introducing this expression in the constraint equation and using (1.24) to express y as a function of x and its derivative, we get a differential equation for x. Since when the Universe isotropises, x and its derivative tend toward zero as Ω diverges, and keeping only the second order terms in x and $\dot{x}$, we find that x asymptotically behaves as $x_0 exp\left[(3-\ell^2)\Omega\right]$ when it tends to vanish. Taking into account the divergence of Ω, we see that our approximation will be justified if $\ell^2 < 3$, which is in accordance with our previous results. One could also recover this result by linearizing the equation (1.24) but we find this demonstration

more rigorous.
Now we examine the case for which ℓ diverges. It implies that the equilibrium points are unbounded. However, since y and z are bounded, we deduce that an equilibrium state can not be reached when ℓ diverges.

In the next section, we discuss about physical meaning of our results.

1.4 Discussion

In this work, we have examined the conditions under which the Universe described by a Generalized Scalar-Tensor theory with a massive scalar field isotropises as well as the asymptotical behaviour of the metric functions by help of the Hamiltonian formalism and dynamical systems theory.
The set of points of the phase space corresponding to stable isotropic states for the Universe is such that the time coordinate Ω and the Hamiltonian diverges ($\Omega \to -\infty$ and $x \to 0$). Then, the functions $\beta_\pm$ describing the anisotropy asymptotically tend toward a constant. We have shown that when ℓ was asymptotically unbounded, an equilibrium state could not be reached. Thus, the isotropy of the Universe is not compatible with the divergence of the quantity $\phi U_\phi U^{-1}(3+2\omega)^{-1/2}$. However it arises when its value belongs to the range $\left[-\sqrt{3},\sqrt{3}\right]$. In this case, the plane $x=0$ contains two equilibrium points corresponding to an isotropic state for the Universe. They are late times attractors in the t time if the Hamiltonian is a positive function. If H is interpreted as an energy, it means that we assume a positive energy for the Universe, which is reasonable. If it is not the case, the isotropisation arises at early times. Moreover, we have shown that as long as $\ell^2 < 3$, the function $x(\Omega)$ asymptotically tended toward $x_0 exp\left[(3-\ell^2)\Omega\right]$. Using (1.16), we see that it corresponds to a power law of the proper times with the exponent ℓ^{-2} if ℓ does not tend toward a vanishing constant, or toward an exponential of t otherwise. These two types of functions represent the only possible attractors when the Universe isotropises. All this can be summarised in the following important result:

A necessary condition for isotropisation of the Generalized Scalar-Tensor theory plus a massive scalar field ϕ, whatever the Brans-Dicke coupling function ω and the potential U considered, will be that $\phi U_\phi U^{-1}(3+2\omega)^{-1/2}$ tends toward a constant ℓ with $\ell^2 < 3$. It arises at late times if the Hamiltonian is positive, at early times otherwise. If $\ell \neq 0$ the metric functions tend toward $t^{\ell^{-2}}$. The Universe is expanding and will be inflationary if $\ell^2 < 1$. If $\ell = 0$, the Universe tends toward a De Sitter model.

Note, that the asymptotical behaviour of the metric functions when isotropisation arises does not depend on initial conditions whereas the epoch of isotropisation, i.e. late or early times, depends on the initial sign of the Hamiltonian. One element is missing in this result: the value of the scalar field when the isotropisation arises, i.e. when $\Omega \to -\infty$. Expressing $\dot{\phi}$ as a function of z and ϕ (see appendix 1.5), and taking z as its value at the equilibrium, $\ell/6$,we get a differential equation for ϕ. It does not describe the scalar field behaviour during the whole Universe evolution, but asymptotically when $\Omega \to -\infty$ and the system approach equilibrium. This equation of the first order can be solved analytically or numerically. This additional result allows to calculate ℓ when isotropisation occurs and completes the main one above. It is written:

The value of the scalar field when the Universe reaches an isotropic equilibrium state is the value of the function ϕ defined by $\dot{\phi} = 2\phi^2 U_\phi (3+2\omega)^{-1} U^{-1}$ when $\Omega \to -\infty$.

Lets examine the relations between these results and others quoted in the literature.

Firstly they are in accordance with the "No Hair Theorem" which states that General Relativity with a cosmological constant isotropises toward a De Sitter model since in this case, $\ell = 0$. It will be true for any form of potential and Brans-Dicke coupling function such that ℓ asymptotically vanishes when $\Omega \to -\infty$ which does not necessary implies that the potential tends toward a constant. As instance, it arises if the Brans-Dicke coupling function diverges faster than $\phi U_\phi U^{-1}$. This generalises the "No Hair Theorem" in the special case of the Bianchi type I model.

Secondly, in [86], it has been shown that all the Bianchi models with an exponential potential $V = e^{k\phi}$ (except the contracting Bianchi type IX model), isotropise at late times when $k^2 < 2$. If $k = 0$, these models tend toward a De Sitter model and toward $t^{2k^{-2}}$ otherwise. If $k^2 > 2$, the Bianchi type I, V, VII and IX models might isotropise at late times. In the present paper, the form of the coupling constant corresponding to the theory studied in [86] is $\sqrt{3+2\omega}\phi^{-1} = \sqrt{2}$. What can we deduce from our results? If we introduce these forms of ω and U in the expression of ℓ, we see that the necessary condition for the isotropisation of the Bianchi type I model will be $k^2 < 6$. Then, the Universe is of De Sitter type if $k = 0$. In the other cases, the metric functions behave as $t^{2k^{-2}}$. The inflation arises when $k^2 < 2$. All these results are in accordance with these of [86] and [85]. However, some differences exist which are not in contradiction with the previous quoted papers: we have shown that Universe might isotropise and is inflationary when $k^2 < 2$ but isotropisation is impossible if $k^2 > 6$. Between these two values, the necessary condition for isotropy is respected but no inflation can occur.

Last, these results are agreed with these found in [160]. In this paper where the Hyperextended Scalar Tensor theory with a potential is studied for the Bianchi type I model, it is shown that the Universe isotropises when $\int Ge^{3\Omega}dt$ tends toward a constant, G being the gravitational coupling function. If in this last expression we choose $G = 1$ and $e^{3\Omega} \rightarrow t^{-3l^{-2}}$, we find that isotropisation arises if $l^2 < 3$, in agreement with the above results.

Lets say few words about the power law potential, $U = \phi^k$. We can show from the asymptotical equation for ϕ defined above, that when $\Omega \rightarrow -\infty$, $\phi \rightarrow +\infty$ if $k < 0$ (if $k > 0$, the scalar field is not real). Then $\ell \rightarrow 0$ and isotropisation systematically leads to an asymptotical De Sitter model.

The result of this paper is not only a necessary condition for the isotropisation of the Universe. We have also derived the asymptotical behaviour of the metric functions and thus some conditions for a late time inflationary behaviour. It is a strong theoretical constraint on the forms of ω and U so that the Universe be physically realistic at late times if we consider that it can be described by a Generalized Scalar-Tensor theory with a massive scalar field in the Bianchi type I model. We have checked the compatibility of our results with these of the important No Hair Theorem and these of Kitada et al and Cooley et al that are here unified in a single condition. To our knowledge, there is no paper mixing the Hamiltonian technique and the dynamical systems theory with so many arbitrary functions. It seems to be a fruitful method in the case studied here mainly because it allows to calculate the equilibrium points as function of the potential and Brans-Dicke function. Then from mathematical constraints on the equilibrium points, we deduce constraints for these undetermined quantities. In future papers, we will see that we get the same type of results when we introduce a perfect fluid and we will extend this method to more general theory such has Hyperextended Scalar Tensor ones or other Bianchi models.

1.5 Appendix

Equation for x

From (1.15) and (1.17), we deduce:

$$\dot{x} = 3R^2y^2x \tag{1.28}$$

Equation for y

By deriving (1.18), we find:

$$\dot{\phi} = 2L_2^{-1}x^{-2}y(\dot{y} - 3R^2y^3) \tag{1.29}$$

with $L_2 = (e^{-6\Omega}U)_\phi$. By using (1.12) and (1.19) to express p_ϕ, we find:

$$\dot{\phi} = 12zL_3^{-1} \tag{1.30}$$

with $L_3 = (3 + 2\omega)^{1/2}\phi^{-1}$ Consequently, (1.29) and (1.30) gives:

$$\dot{y} = 6L_2L_3^{-1}x^2y^{-1}z + 3R^2y^3 \tag{1.31}$$

Equation for z

by using (1.19) to express p_ϕ and by deriving this expression, we get:

$$\dot{p_\phi} = L_3x^{-1}\dot{z} - 3R^2L_3x^{-1}y^2z - 12\phi^{-1}x^{-1}z^2 + 12L_4L_3^{-2}\phi^{-2}x^{-1}z^2 \tag{1.32}$$

with $L_4 = \omega_\phi$. From (1.14) and by using the fact that $U^{-1}U_\phi = L_2x^2y^{-2} + 1/2L_3z^{-1}$, it comes:

$$\dot{p_\phi} = -12\phi^{-1}x^{-1}z^2 + 12L_4L_3^{-2}\phi^{-2}x^{-1}z^2 - \frac{R^2}{2}L_2x - \frac{R^2}{4}L_3x^{-1}y^2z^{-1} \tag{1.33}$$

From the equations (1.32) and (1.33), we derive:

$$\dot{z} = 3R^2y^2z - \frac{R^2}{2}L_2L_3^{-1}x^2 - \frac{R^2}{4}y^2z^{-1} \tag{1.34}$$

Before getting the equation (1.24) to (1.26), we have to evaluate the term $L_2L_3^{-1}x^2y^{-2}$. After few calculations, we find $-(2z)^{-1} + \phi U^{-1}U_\phi(3 + 2\omega)^{-1/2}$.

Chapitre 2

Isotropisation of the minimally coupled scalar-tensor theory with a massive scalar field and a perfect fluid in the Bianchi type I model

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Abstract

We look for necessary conditions such that minimally coupled scalar-tensor theory with a massive scalar field and a perfect fluid in the Bianchi type I model isotropises. Then we derive the dynamical asymptotical properties of the Universe.

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2.1 Introduction

In this work, we will study the isotropisation of a minimally coupled scalar tensor theory with a massive scalar field and a perfect fluid in the Bianchi type I model.
Lets give some reasons about our choice for this geometrical framework. Although our present Universe seems in agreement with the cosmological principle, it could be necessary to partly release it for the early times if we want to explain the formation of large-scale structures. Moreover, the observations of the cosmic microwave background by COBE show some small inhomogeneities. These observational facts lead us to assume a more general geometry than this of the isotropic models i.e. the Friedman-Lemaitre-Robertson-Walker (FLRW) ones. The simplest generalisation is to leave the isotropy hypothesis and to consider the Bianchi models. About the nine Bianchi models, some of them accept FLRW models as exact solutions. Hence, Bianchi type I models contain the flat FLRW ones , the Bianchi type V ones contain the open FLRW ones, and the Bianchi type IX, the closed ones. At the present time, despite some more and more powerful observational tools, we do not know in which type of Universe we live. However, Boomerang experiment favours a flat Universe[37] and the same conclusion could be drawn from the presence of inflation[9, 10] although this phenomenon has to be confirmed by deepest observations and could be compatible with other types of geometries[164]. All these facts justify the interest of the Bianchi type I model.

As a physical framework, we have chosen to study a scalar-tensor theory minimally coupled to a massive scalar field ϕ. The geometrical part of its Lagrangian writes as this of the General Relativity, which describes with a high precision the local dynamics of our Universe for the weak fields. Taking into accounts one or several scalar fields could be one of the key for a theory able to explain the physics of the early times.

They are predicted by unification theories whose low energy limit could be a scalar-tensor theory. They also appear during dimensional reduction of Kaluza-Klein type theories. In addition, we will also assume that ϕ is a massive field. The reason is that inflation could be the consequence of the presence of a cosmological constant whose currently observed value and particle physics predicted value differ from 120 orders of magnitude: this is the so called cosmological problem. To explain this huge discrepancy, a solution could be to consider that the cosmological constant is in fact a variable potential U representing the coupling of ϕ with itself. Moreover, we will consider a Brans-Dicke coupling function ω between the scalar field and the metric. The theory thus described has been studied in [105]. Here, we will generalise it by adding a perfect fluid. Associating a scalar field and a perfect fluid could be a way to explain the nature of dark matter, if the first one plays the same dynamical role as the second one as suggested by the quintessence or tracking models[165, 136]. In the quintessence model, the scalar field slowly rolling down its potential such that the ratio of its pressure and energy density, w, be a constant belonging to the range $[-1{,}0]$. One problem of the quintessence model is the cosmic coincidence problem: why the present energy density of the scalar field would be of the same order as this of the matter energy density. One possible solution to this question could be to consider a special form of quintessence, called tracker model, for which w is time varying, and works like an attractor solution. Thus, late times cosmology would be independent of the early conditions.

From an observational point of view, the standard model that seems to emerge today is the same as this described above with $\omega = 0$ and $U = cte$. However, this particular one is far from being satisfactory (as instance, it does not solve the cosmological problem). Hence, more general theories leaving ω and U undetermined have to be studied and some efforts are currently done to try and guess what could be some limitation on their forms and values. As instance, in [41], it is shown how from the observations one can determine the Lagrangian of a scalar tensor theory. In [166], it is demonstrated that scalar tensor theories can be compatible both with primordial nucleosynthesis and solar-system experiments with cosmological models very different from the FLRW ones. In [167], it is shown how new bounds on ω could be derived from future space gravitational wave interferometers, thus allowing to test scalar tensor gravity. All these works are related to observational cosmologies and aim to derive some satisfactory limits on ω and U as we try to do it from a theoretical ground in the present paper.
Mathematically we will study the isotropisation of the Universe in the same way as in [105], i.e. by associating the Hamiltonian formalism of Arnowitt, Deser and Misner (ADM)[78, 77] which allows getting first order dynamical system equations with the dynamical system methods[25]. Our goal will be to determine the necessary conditions for the isotropisation of the theory described above and the asymptotical dynamical behaviours of the Universe at late times. The plan of this work is the following. In the second section, we establish the field equations of the Hamiltonian formalism. In the third section, we analyse their dynamics. In the fourth section, we discuss the physical meaning of our results.

2.2 Field equations

In this section, we will calculate the field equations of the minimally coupled scalar-tensor theory with a massive scalar field and a perfect fluid. The action is written:

$$S = (16\pi)^{-1}\int \left[R - (3/2 + \omega(\phi))\phi^{,\mu}\phi_{,\mu}\phi^2 - U(\phi) + 16\pi c^4 L_m\right]\sqrt{-g}d^4x \quad (2.1)$$

with ϕ the scalar field, ω the coupling between the scalar field and the metric, U the potential and L_m the Lagrangian density of the matter. We will consider a perfect fluid with an equation of state $p = (\gamma - 1)\rho$, p and ρ being respectively the pressure and the density of the fluid. For $\gamma = 0$, $\gamma = 1$ and $\gamma = 4/3$ we get respectively the equation of state describing the vacuum energy, a dust and radiation fluid. As the first case can be assimilated to the presence of a cosmological constant already studied in [105], we will not consider it and will assume that $\gamma \in [1{,}2]$. We will use the following form of the metric:

$$ds^2 = -(N^2 - N_iN^i)d\Omega^2 + 2N_id\Omega\omega^i + R_0^2 g_{ij}\omega^i\omega^j \quad (2.2)$$

The ω^i are the 1-forms defining the homogeneous space of the Bianchi type I model and the g_{ij} are the metric functions. To derive the expression of the ADM Hamiltonian, we proceed in the same way as [78] or [42]. We rewrite the metric functions by using the Misner parameterisation[24]:

$$\begin{aligned} g_{11} &= e^{-2\Omega+\beta_+ + \sqrt{3}\beta_-} \\ g_{22} &= e^{-2\Omega+\beta_+ - \sqrt{3}\beta_-} \end{aligned}$$

$$g_{33} = e^{-2\Omega-2\beta_+}$$

The $\beta_\pm$ functions describe the anisotropy of the Universe and Ω its isotropic part. We can derive the field and the Klein-Gordon equations of the Lagrangian formulation by varying the action with respect to the metric functions and the scalar field. Then using the Bianchi identities and the Klein-Gordon equation, we can find the conservation law for the energy impulsion of the perfect fluid, $T^{0\mu}_{;\mu} = 0$, and thus the well known relation between the energy density and the 3-volume V of the Universe: $\rho = \rho_0 V^{-\gamma}$, ρ_0 being an integration constant and $V = (\prod_i g_{ii})^{1/2} = e^{-3\Omega}$. To find the Hamiltonian of the ADM formalism, we have to express the action with the variables β_+, β_- and their conjugate momentum p_+ and p_-. Then, we vary it with respect to N which plays the role of a Lagrange multiplier. We get a constraint equation from which we can derive the Hamiltonian. The reader interested by full details of its derivation when no perfect fluid is present can find it in [105]. Finally, we get:

$$H^2 = p_+^2 + p_-^2 + 12\frac{p_\phi^2\phi^2}{3+2\omega} + 24\pi^2 R_0^6 e^{-6\Omega}U + \delta e^{3(\gamma-2)\Omega} \tag{2.3}$$

p_ϕ is the conjugate momentum of the scalar field and δ a constant equal to $(\gamma-1)\rho_0$. It is a positive constant when $\gamma \in [1{,}2]$ and the energy density of the perfect fluid is positive. We will assume it is the case. From (2.3), we derive the Hamilton's equations:

$$\dot\beta_\pm = \frac{\partial H}{\partial p_\pm} = \frac{p_\pm}{H} \tag{2.4}$$

$$\dot\phi = \frac{\partial H}{\partial p_\phi} = \frac{12\phi^2 p_\phi}{(3+2\omega)H} \tag{2.5}$$

$$\dot p_\pm = -\frac{\partial H}{\partial \beta_\pm} = 0 \tag{2.6}$$

$$\dot p_\phi = -\frac{\partial H}{\partial \phi} = -12\frac{\phi p_\phi^2}{(3+2\omega)H} + 12\frac{\omega_\phi \phi^2 p_\phi^2}{(3+2\omega)^2 H} - 12\pi^2 R_0^6 \frac{e^{-6\Omega}U_\phi}{H} \tag{2.7}$$

$$\dot H = \frac{dH}{d\Omega} = \frac{\partial H}{\partial \Omega} = -72\pi^2 R_0^6 \frac{e^{-6\Omega}U}{H} + 3/2\delta(\gamma-2)\frac{e^{3(\gamma-2)\Omega}}{H} \tag{2.8}$$

A dot means a derivative with respect to Ω. If we compare them with these got when no perfect fluid is present[105], we remark that only the equation (2.8) is modified. We will choose $N^i = 0$ and then we calculate that N [78][77] can be written[1]:

$$N = \frac{12\pi R_0^3 e^{-3\Omega}}{H} \tag{2.9}$$

The relation between the Ω time and the proper time t being $dt = -Nd\Omega$, we deduce that t is a decreasing function of Ω for all positive Hamiltonian.

2.3 Isotropisation conditions and asymptotical behaviours

In the first subsection, we rewrite the field equations with new normalised variables. In the second one, we study mathematically the system thus obtained.

2.3.1 Rewriting of the field equations with normalised variables

The equations (2.3-2.8) form a first order system that we wish to rewrite with the following variables:

$$x = H^{-1} \tag{2.10}$$

$$y = e^{-3\Omega}\sqrt{U}H^{-1} \tag{2.11}$$

1. In few words, this result can be recovered in the following way. We first rewrite the action under a Hamiltonian form, i.e. $S = \int (g_{ij}\frac{\partial \Pi_{ij}}{\partial t} - NH - N_i H_i)d^4x$, Π_{ij} being the conjugate momentum of the metric functions g_{ij}, N and N_i the lapse and shift functions, H and H_i the super Hamiltonian and super momentum. Then, by varying the expression thus obtained with respect to Π_{ij}, we derive an expression for $\partial g_{ij}/\partial t$. Developing $-1/2(-g)^{-1/2}\delta g/\delta\Omega$ as a functions of the g_{ij} and using the above mentioned expression, we find (2.9).

$$z = p_\phi \phi (3 + 2\omega)^{-1/2} H^{-1} \tag{2.12}$$

They are independent each other's since the first one depends on H, the second one on H and ϕ and the third one on H, ϕ and p_ϕ. The forms of y and z show that the potentials U have to be positive and the Brans-Dicke coupling function ω must be larger than $-3/2$ so that the variables be real. These are usual hypothesis in cosmology. The constraint (2.3) is then written:

$$p^2 x^2 + R^2 y^2 + 12 z^2 + k^2 = 1 \tag{2.13}$$

where we have put to simplify the equations $k^2 = \delta e^{3(\gamma-2)\Omega} H^{-2} = \delta x^\gamma y^{2-\gamma} U^{\gamma/2-1}$. The constants p and R are defined by $p^2 = p_+^2 + p_-^2$ and $R^2 = 24\pi^2 R_0^6$. Rewriting the equations (2.5), (2.7) and (2.8), we get:

$$\dot{x} = 3R^2 y^2 x - 3/2(\gamma - 2) k^2 x \tag{2.14}$$

$$\dot{y} = y(6\ell z + 3R^2 y^2 - 3) - 3/2(\gamma - 2) k^2 y \tag{2.15}$$

$$\dot{z} = y^2(3Rz - R^2/2\ell) - 3/2(\gamma - 2) k^2 z \tag{2.16}$$

with $\ell = \phi U_\phi U^{-1} (3 + 2\omega)^{-1/2}$. From the first one, we deduce that x is a monotonic function of constant sign and then that no homoclinic orbit are allowed.
In the 2.4, we show that even if $\delta < 0$, i.e. if we consider negative energy density, the equilibrium is not compatible with the divergence of k.

2.3.2 Mathematical study of the first order system equations

In the first subsection, we examine the values of x and k allowing the isotropisation. In the second one, we look for the equilibrium points corresponding to a stable isotropic state for the Universe.

Values of x and k compatible with the isotropisation

Assuming that late times correspond to the divergence of the proper time t, an isotropic and stable state is such that $\beta_\pm$ tend toward some constants with $d\beta_\pm/dt \to 0$. By using the expression for the lapse function, we calculate that it happens only when $\Omega \to -\infty$. The Universe is then expanding. It corresponds to late times epoch when the Hamiltonian is positive and justify our assumption on the asymptotical value of t since there is no physical reason such that the diverging expansion of the Universe take place for a finite value of the proper time. Since in the same time, $\beta_\pm$ should tend toward some constants, we should also have $\dot{\beta}_\pm \to 0$, i.e. $x \to 0$. Consequently, a stable isotropic state can be reached only in the plane $x = 0$ of the phase space when Ω diverges negatively. We will have to check if these two conditions do not excluded each others.

What about the value of k? From the constraint, we see that $k^2 \leq 1$. Then, by considering the expression $k^2 = \delta H^{-2} e^{3(\gamma-2)\Omega}$ and the equation for $\dot{H}H$ issued from (2.8), we deduce that:

- k^2 will tend to vanish when $U > V^{-\gamma}$.
- k^2 will tend toward a constant different from zero when $x = 0$ if the Hamiltonian tends toward $H = H_0 e^{3/2(\gamma-2)\Omega}$. Then $k^2 \to \delta H_0^{-2}$ and $U \propto V^{-\gamma}$.
- k^2 will tend toward 1 for any potential such that $U < V^{-\gamma}$.

The first case corresponds to an asymptotically dominated scalar field Universe. It has already been studied in [105]. It will concern any potential tending toward a constant or diverging since the 3-volume will diverge when isotropisation arises and then $V^{-\gamma} \to 0$. The second case corresponds to a potential behaving asymptotically as the energy density of the perfect fluid, $U \propto \rho$. It is different from trackers solutions which are such that asymptotically $\rho_\phi \propto \rho$, ρ_ϕ being the scalar field energy density: then, it is the scalar field energy density that mimics this of the perfect fluid and $U \leq \rho$. However, in both cases, the metric functions behave asymptotically as the Universe was filled only with a perfect fluid whose density is increased by the presence of the scalar field. Hence, when $U \propto V^{-\gamma}$, dark matter and coincidence problems could be explained in the same way as trackers solutions try to do it. For this reason, we will name these solutions "trackers like" solutions.

The third case corresponds to a Universe dominated by the perfect fluid. Effectively, if we use a Lagrangian formulation to rewrite the field equations, we remark that when $U < V^{-\gamma}$, in the space field equations we can neglect the potential regarding the term of the perfect fluid. Thus, the asymptotical solutions of the metric function will take the same form as these of a theory without scalar field. However, in the constraint equation, the scalar field is not negligible. We will see below that this case always implies that ℓ diverges, that is also possible in the previous case.

In the following subsection, only the non asymptotically scalar field dominated cases that contain the trackers solutions, will be studied, the results of the scalar field dominated case being identical to these of [105].

Isotropic stable equilibrium states for non asymptotically scalar fi eld dominated Universe

We find four equilibrium points in the plane $x = 0$, with $k \neq 0$ and respecting the constraint if $k^2 = 1 - 3\gamma(2\ell^2)^{-1}$. This last condition is not a fine-tuning. In fact, the expression for k contains the integration constant δ which is hence determined by the constraint equation. As evoked above, it shows that when $\ell \to \infty$, $k \to 1$. Moreover, since k^2 and δ are positive, we derive that $\ell^2 \notin [0,3/2\gamma]$ and thus eliminate two of the equilibrium points which are not real under this condition. The two remaining ones are then defined by $(y,z) = (\pm(2R\ell)^{-1}\sqrt{3\gamma(2-\gamma)},(4\ell)^{-1}\gamma)$. They are real as long as $\gamma < 2$. The equilibrium states taking place in $\Omega \to -\infty$, they are sources for the Ω time and sinks for the proper times t when the Hamiltonian, which can be assimilated to an energy, is positive. Linearising the equation (2.14) in the neighbourhood of the equilibrium points allows us to calculate $x(\Omega)$ and, using the relation $dt = -Nd\Omega$, we get that asymptotically $e^{-\Omega}$ tends toward $t^{\frac{2}{3}\gamma^{-1}}$ whatever ℓ. It follows that if the potential behaves like the energy density of the perfect fluid, near the equilibrium, both tend toward zero as t^{-2}. We have also checked that $x(\Omega) \to 0$ when $\Omega \to -\infty$, thus showing the compatibility of these two limits necessary for isotropisation.
In the next section, we discuss about physical meaning of these results, compare them with other papers and make some applications.

2.4 Discussion

In this paper, we have considered the isotropisation of a scalar-tensor theory minimally coupled to a massive scalar field with a perfect fluid for the Bianchi type I model. The following discussion is divided in three parts. The first one contains the set of results of this work, the second one some comparisons with other papers and the third one, some applications concerning the most studied forms of potentials.

We have seen that three cases can be distinguished depending on the fact that the potential is larger, smaller or behaves in the same way as the energy density of the perfect fluid. For the first case, we recall the result we have got in [105], adapting its terms to the present paper:

Asymptotically dominated scalar field Universe
When the potential is asymptotically larger than the energy density of the perfect fluid, a necessary condition for isotropisation of the minimally coupled scalar field with a massive scalar field ϕ and a perfect fluid, whatever the Brans-Dicke coupling function ω and the potential U considered, will be that $\phi U_\phi U^{-1}(3+2\omega)^{-1/2}$ tends toward a constant ℓ with $\ell^2 < 3$. It arises at late times if the Hamiltonian is positive, at early times otherwise. If $\ell \neq 0$ the metric functions tend toward $t^{\ell^{-2}}$. The Universe is expanding and will be inflationary if $\ell^2 < 1$. If $\ell = 0$, it tends toward a De Sitter model.

It includes all the diverging potentials or these tending toward a cosmological constant at late times as it could be the case for our present Universe. The first new result of this study concerns the potentials that mimic asymptotically the energy density ρ of the perfect fluid. In this case, we can consider that their effect is equivalent to increase ρ.

Isotropisation of trackers like theories:
When the potential asymptotically behaves like the positive energy density of the perfect fluid with an equation of state $p = (\gamma-1)\rho$ and $\gamma \geq 1$, necessary conditions for isotropisation will be that $\phi^2 U_\phi^2 U^{-2}(3+2\omega)^{-1}$ tends toward a constant ℓ^2 larger that $\frac{3}{2}\gamma$ and $\gamma < 2$. The isotropisation always arises at late(early) times when the Hamiltonian is positive(negative). Then the metric functions tend toward the attractor $t^{\frac{2}{3}\gamma^{-1}}$, the Universe is expanding, non-inflationary and the potential and density ρ tend toward zero as t^{-2}.

Finally, the last case concerns a theory asymptotically dominated by the matter. Then, the potential U is negligible regarding the energy density ρ:

Asymptotically matter dominated Universe:
When the potential is asymptotically smaller than the energy density of the perfect fluid, the necessary condi-

tions for isotropisation are the same as for trackers like theories but the quantity $\phi^2 U_\phi^2 U^{-2}(3+2\omega)^{-1}$ is always diverging.

To complete these two last results, we need to determinate the asymptotical value of ϕ near the equilibrium. From it, we will be able to determine the asymptotical value of the potential relative to $V^{-\gamma}$ and the constant ℓ. Using the equation (2.5) and writing it near the equilibrium, we deduce that:

Asymptotical behaviour of the scalar field near the equilibrium:
The asymptotical behaviour of the scalar field near the equilibrium state when it does not dominate the Universe is this of the function ϕ defined by the differential equation $\dot\phi = 3\gamma U U_\phi^{-1}$ when $\Omega \to -\infty$.

Note that in [105], it has been shown that the corresponding equation for the scalar field when it dominates the Universe is $\dot\phi = 2\phi^2 U_\phi (3+2\omega)^{-1} U^{-1}$. All these results are independent from an initial state for the Universe except the sign of H which have to be initially positive so that the isotropisation take place at late times. Remark also that they can not be applied to a theory without potential since then the variable y and thus ℓ is not defined.

In this second part, we compare our results related to the non-asymptotically dominated scalar field Universe with these of other papers.
Hence, in [168] is studied what is called the "Quintessential adjustment of the cosmological constant". The quintessence phenomenon is considered for an FLRW model and leads naturally to a vanishing potential. It is what we observe here, the quintessential solutions being such that $U \leq V^{-\gamma}$, since then necessary conditions for isotropisation imply that *the potential tends toward zero at the most as* t^{-2}. It could thus solve the cosmological constant problem.
In [163] where the General Relativity with a perfect fluid and a quintessential matter is studied for a flat isotropic model, it has been demonstrated that *the solving of the coincidence problem was not compatible with inflation*. This result is here generalised to any isotropising Bianchi type I model whatever ω and U since for inflation being present at late time we would need that $\gamma < 2/3$, which is not the case for ordinary matter.
In [62], it has been shown that scalar-tensor theory with a potential and a perfect fluid can *have as late time attractor the General Relativity*. It corresponds to what we have found in this work since when necessary conditions for isotropy are respected the metric functions asymptotically tend toward $t^{\frac{2}{3}\gamma^{-1}}$. Hence, when $\gamma = 1$, we have a dust fluid and the Universe tends toward an Einstein De-Sitter one with $g_{ij} \to t^{2/3}$ as usually found when we consider this kind of matter. The same remark is valid when the equation of state represents a radiative fluid with $\gamma = 4/3$. Then, the Universe tends toward a Tolman one with $g_{ij} \to t^{1/2}$.
In [160], Hyperextended scalar tensor theories (HST) with a potential for the Bianchi type I model are studied. HST has the same action as (2.1) but with a gravitational function depending on the scalar field[35]. Some results of this last study are not changed by the presence of a perfect fluid. Especially it has been shown that the Universe isotropises when $Ge^{3\Omega}$ tends toward a constant. Applied to the present paper, it gives that γ have to be smaller than two which is the reality condition for the equilibrium points.
If we consider General Relativity with only a perfect fluid, it is known that the Bianchi type I model isotropises. In the present case, we observe that the presence of a scalar field add a necessary constraint, related to the range of value of the constant ℓ, such that isotropisation might arise. We also note that the asymptotical behaviour of the metric functions does not depends on ℓ contrary to the case for which the scalar field dominates.
Last, from a mathematical point of view, the main difference between the case where the matter is absent[105] or does not dominate at late times and the case of trackers like theories or matter dominated theories comes from the reality condition that selects the equilibrium points. In any circumstances, the field equations written with new variables admit four equilibrium points that we will name E_1, E_2, E_3 and E_4. In the first case, reality condition selects the two first points and ℓ belongs to a closed interval such that $\ell^2 < 3$. In the second case, the two last points are selected and ℓ belongs to an open interval such that $\ell^2 > \frac{3}{2}\gamma$.

In what follows, we are going to study two well-known theories defined by $\sqrt{3+2\omega}\phi^{-1} = \sqrt{2}$ and some exponential and power laws potential. They have mainly been considered for FLRW models and most of the results we will get will not be new. However they will permit us to test these of the present paper and to show that the formalism we use allow unifying them. To study each of these theories, we will proceed in four steps:

1. We calculate the asymptotical value of the scalar field if we assume that $U \leq V^{-\gamma}$ or $U > V^{-\gamma}$.
2. We respectively deduce the conditions such that $U \leq V^{-\gamma}$ or $U > V^{-\gamma}$.
3. We respectively deduce the conditions on ℓ such that the Universe isotropise.
4. We compare if needed, the two sets of conditions to check their compatibility.

The first theory we want to study is defined by $\sqrt{3+2\omega}\phi^{-1} = \sqrt{2}$ and $U = e^{k\phi}$. In [86], it is examined without a perfect fluid and demonstrated that isotropisation arises for $k^2 < 2$. In [105] it is shown that it can not happen when $k^2 > 6$. If we suppose that the scalar field does not dominate at late times, we calculate that near the equilibrium ϕ behaves as $3\gamma k^{-1}\Omega$. Thus whatever k, the asymptotical behaviour of the potential is the same as this of the perfect fluid energy density and hence the solution will be trackers like solutions. The necessary condition for isotropisation related to ℓ is then written $k^2 > 3\gamma$. The calculus of $d\phi/dt$ shows that this derivative is of the same order or smaller than the potential and thus this solution is a true trackers one as usually considered in the literature. If now we assume that the scalar field asymptotically dominates the Universe and that we use the results of [105], the asymptotical form of ϕ shows that our assumption is true only if $k^2 < 3\gamma$. The necessary condition for isotropisation given by the limit on ℓ is then $k^2 < 6$. Since $\gamma < 2$, only the first inequality on k have to be taken into account. Hence, when the scalar field asymptotically dominates the Universe, the necessary condition for isotropisation is satisfied. To summarise, if $k^2 > 3\gamma$, necessary conditions for isotropisation of the Universe toward a tracker solution such that $e^{-\Omega} \rightarrow t^{\frac{2}{3}\gamma^{-1}}$ are respected whereas if $k^2 < 3\gamma$, necessary conditions are respected such that it be able to isotropise toward a dominated scalar field Universe with $e^{-\Omega} \rightarrow t^{2k^{-2}}$[86, 105]. These results have been derived in [111] for the FLRW models. However, in this last paper, a stable trackers solution corresponding to $k^2 > 6$ have also been found. It does not exist in this work since then the isotropisation would be impossible.

The second theory we wish to consider is defined by $\sqrt{3+2\omega}\phi^{-1} = \sqrt{2}$ and $U = \phi^k$. If we assume that the late times Universe is not asymptotically dominated by the scalar field, we find that ϕ will tend toward $\phi_0 e^{3\gamma k^{-1}\Omega}$, ϕ_0 being an integration constant[2]. If $k < 0$, ℓ tends to vanish and isotropisation is not possible. If $k > 0$, ℓ diverges and necessary conditions for isotropisation are respected. Thus we deduce that the perfect fluid will asymptotically dominate this solution, hence confirming our assumption. If now we suppose that $U > V^{-\gamma}$, the calculus of the scalar field confirm it and, as shown in [105], the theory is able to isotropise toward a De-Sitter model asymptotically dominated by the scalar field when $k < 0$. These results are in accordance with these found in [169] where it has been shown that for $k < 0$, the solution is asymptotically dominated by the scalar field whereas when $k > 0$, it is matter dominated.

Some particular cases of minimally coupled scalar tensor theories with a massive scalar field and a perfect fluid has already been studied in the literature. Here we have made an attempt to derive some necessary conditions such that an asymptotically isotropic stable state be reached by the Universe at late times whatever U and ω and we have then studied its dynamical behaviour. Using these results, we have made two applications and checked their consistency with previous works. In a future paper, we hope to apply the mathematical methods of this work to the Hyperextended Scalar Tensor theory for which the gravitational function varies with the scalar field.

Appendix: Divergence of k

Since we have chosen to consider some positive energy densities for the perfect fluid with moreover $\gamma \in [1,2]$, then k^2 is positive and thus from the constraint, we deduce that the divergence of k is excluded. However, in what follows, we will consider that $\delta < 0$. For the constraint it is equivalent to write it as $p^2x^2 + Ry^2 + 12z^2 - k^2 = 1$ or to keep the same form as (2.13) but with $k^2 < 0$. We will consider this last possibility. Then, k can diverge but we wish to show that it is not compatible with an equilibrium state.

In a general manner, when $x \rightarrow 0$, the plane where all the isotropic stable states are present as shown in subsection 2.3.2, we have the following relations:

$$k^2 \rightarrow 1 - Ry^2 - 12z^2 \tag{2.17}$$

and then:

$$\dot{y} \rightarrow y(6\ell z + 3Ry^2 - 3 - 3/2(\gamma - 2)(1 - Ry^2 - 12z^2)) \tag{2.18}$$

$$\dot{z} \rightarrow 3Ry^2 z - R/2\ell y^2 - 3/2(\gamma - 2)z(1 - Ry^2 - 12z^2) \tag{2.19}$$

2. This constant does not appear in the previous application since it is asymptotically negligible.

The expressions (2.18) and (2.19) have to tend toward zero to reach equilibrium. For the first one, it will arise if $y \to 0$ or $6\ell z + 3Ry^2 - 3 - 3/2(\gamma - 2)(1 - Ry^2 - 12z^2) \to 0$. Let's study these two possibilities.

Case 1: $y \to 0$ and (2.18)$\to 0$

Then, (2.17) implies that z diverges and (2.18) that yz^2 tends toward zero. Applying these two limits to (2.19), we deduce that $\dot{z} \to z^3$ and thus diverges, preventing the equilibrium. This reasoning is also valid when ℓ diverges.

Case 2: $6\ell z + 3Ry^2 - 3 - 3/2(\gamma - 2)(1 - Ry^2 - 12z^2) \to 0$ and (2.18)$\to 0$

It means that:

$$y^2 \to \left[-6\ell z + 3 + 3/2(\gamma - 2)(1 - 12z^2)\right](3/2R\gamma)^{-1} \quad (2.20)$$

$$k^2 \to 1 - \left[-6\ell z + 3 + 3/2(\gamma - 2)(1 - 12z^2)\right](3/2\gamma)^{-1} - 12z^2 \quad (2.21)$$

By putting this last expression in (2.19), we get an expression of $\dot{z}$ as a function of z. An equilibrium point can then be reached only for a finite value of z. But (2.20) shows that y will tend toward a constant. Thus, it will be the same for k which contradicts the fact that it becomes infinite. It is the same if ℓ diverges.

Consequently, a diverging value of k is not compatible with an isotropic state for the Universe at late times whatever the sign of δ.

Chapitre 3

Isotropisation of Bianchi class A models with curvature for a minimally coupled scalar tensor theory

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Abstract

We look for necessary isotropisation conditions of Bianchi class A models with curvature in presence of a massive and minimally coupled scalar field when a function ℓ of the scalar field tends to a constant, diverges monotonically or with sufficiently small oscillations. Isotropisation leads the metric functions to tend to a power or exponential law of the proper time t and the potential respectively to vanish as t^{-2} or to a constant. Moreover, isotropisation always requires late time accelerated expansion and flatness of the Universe.

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3.1 Introduction

In this paper we study isotropisation of Bianchi class A models with curvature when a minimally coupled and massive scalar field ϕ is considered.

Locally, General Relativity (GR) with a perfect fluid seems a good description of our Universe. At cosmological scale, accelerated dynamical expansion is observed[9, 10] and additional fields are required to explain it. Among them, a scalar field seems a good alternative although it is not the only one: higher order theories[170, 171] or dissipative fluid[163, 172] also give birth to inflationary behaviour. Scalar fields are required by standard model for elementary particles as well as by unification theories for which, for instance, compactification schemes[113, 26] are considered. These last theories also give a natural order of magnitude for the cosmological constant[173] at early times which may be 55 to 120 orders of magnitude bigger than its present observed value: this is the so-called cosmological constant problem. A solution is to consider that this ”constant” varies across the Universe history. A massive scalar field is then an interesting possibility to simulate such a mechanism. All these elements show the interest of a minimally coupled scalar-tensor theory with a Brans-Dicke coupling function ω and a potential U depending on the scalar field ϕ.

What about the geometrical framework of this paper? FLRW models geometrically describe the observed homogeneity and isotropy of our Universe. However they are very special ones among the set of all possible models and do not allow to explain the observed large-scale structures. Moreover, at early times, before the decoupling between matter and radiation, we have no indication about Universe's geometry. Was it as so symmetric as the FLRW models imply? Thus, it seems interesting to generalise them by only keeping their spatial homogeneity property. Bianchi models describe anisotropic cosmological models and may allow to understand the process leading to an isotropic Universe. The most studied Bianchi models are those containing the FLRW solutions[108], i.e. the types I, V, $VII_{0,h}$ and IX. We have examined the Bianchi type I model in [105]. Here, we will be interested in the Bianchi class A models with curvature.

Our goal is to look for necessary conditions allowing the isotropisation of Bianchi class A models with curvature when a minimally coupled and massive scalar tensor theory is considered. We will then deduce the common asymptotical behaviour of the metric functions when isotropisation is reached and compare our results with those obtained for the Bianchi type I model[105]. From a technical point of view, we will use the methods of [105]: we will get the field equations from ADM Hamiltonian formalism[78, 77] and rewrite them with a new set of variables. Then we will look for equilibrium points corresponding to isotropic stable states.

A large amount of work has been done on equilibrium states of Bianchi models. Wainwright, Ellis and collaborators have studied equilibrium points of homogeneous models for General Relativity with perfect fluid, tilted or not and found asymptotically isotropic solutions. A good summary of their work is [25]. They use Hubble-normalized variables to study the dynamics of Einstein field equations. The normalisation factor is the Hubble parameter and the equation allowing to show that variables are normalized is the generalized Friedman equation. In this paper we will also consider normalized variables but we will use the Hamiltonian as normalisation factor. The expression for Hamiltonian will be the constraint from which we deduce that variables are normalized. More recently Barrow and Kodama[174, 175] have examined the influence of topology on isotropy and flatness of the Universe. They have shown that "the topology of the Universe can impose significant restrictions upon the type of anisotropies it can sustain". We will not consider topology in this work but these results are really interesting from the point of view of relations between dynamics and topology which has also been examined by Ashtekar and Samuel[176].

The plane of the paper is the following. The second section will be parsed into three subsections. The first one will be devoted to the Bianchi type II model, the second one to the Bianchi types VI_0 and VII_0 and the third one to the Bianchi type $VIII$ and IX models. Each of these subsections will be divided into two subsections devoted to the field equations and the study of the equilibrium points. We will discuss the physical meaning of our results in section 3.3.

3.2 Mathematical study of isotropisation for class A Bianchi models

We begin calculating the Hamiltonian field equations. The Lagrangian of the minimally coupled scalar-tensor theory is given by:

$$S = (16\pi)^{-1}\int \left[R - (3/2+\omega(\phi))\phi^{,\mu}\phi_{,\mu}\phi^{-2} - U(\phi)\right]\sqrt{-g}d^4x \tag{3.1}$$

Although it may be more natural to redefine ϕ so that the kinetic term takes a standard form $\phi^{,\mu}\phi_{,\mu}$, we prefer considering an unspecified Brans-Dicke coupling function such that our results be valid for any form of $\omega(\phi)$ even when it is analytically impossible to get $\phi(\omega)$. The general form of the metric for Bianchi models is written:

$$ds^2 = -(N^2 - N_iN^i)d\Omega^2 + 2N_id\Omega\omega^i + R_0^2 g_{ij}\omega^i\omega^j \tag{3.2}$$

The ω_i are the 1-forms defining each Bianchi model. N and N^i are respectively the lapse and shift functions. To find the ADM Hamiltonian corresponding to the action (3.1), we proceed as in [78] and [77]. We rewrite the action as follows:

$$S = (16\pi)^{-1}\int (\Pi^{ij}\frac{\partial g_{ij}}{\partial t} + \Pi^{\phi}\frac{\partial \phi}{\partial t} - NC^0 - N_iC^i)d^4x \tag{3.3}$$

The Π_{ij} and Π_ϕ are respectively the conjugate momenta of the metric functions and scalar field. The lapse and shift functions now play the role of Lagrange multipliers. By varying (3.3) with respect to N and N_i, we get the constraints $C^0 = 0$ and $C^i = 0$ with:

$$C^0 = -\sqrt{^{(3)}g}\,^{(3)}R - \frac{1}{\sqrt{^{(3)}g}}(\frac{1}{2}(\Pi_k^k)^2 - \Pi^{ij}\Pi_{ij}) + \frac{1}{\sqrt{^{(3)}g}}\frac{\Pi_\phi^2\phi^2}{6+4\omega} + \sqrt{^{(3)}g}U(\phi) \tag{3.4}$$

$$C^i = \Pi^{ij}_{|j} \tag{3.5}$$

We rewrite the metric functions g_{ij} as $e^{-2\Omega+2\beta_{ij}}$. It means that Ω stands for the isotropic part of the metric and β_{ij} for the anisotropic parts. Then, using Misner parameterisation [79]:

$$p^i_k = 2\pi\Pi^i_k - 2/3\pi\delta^i_k\Pi^l_l \tag{3.6}$$

$$6p_{ij} = diag(p_+ + \sqrt{3}p_-, p_+ - \sqrt{3}p_-, -2p_+) \tag{3.7}$$

$$\beta_{ij} = diag(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+) \tag{3.8}$$

and rewriting the Hamiltonian as $H = 2\pi\Pi_k^k$, from the expression (3.4) and the constraint $C^0 = 0$, we get:

$$H^2 = p_+^2 + p_-^2 + 12\frac{p_\phi^2\phi^2}{3+2\omega} + 24\pi^2 R_0^6 e^{-6\Omega}U + V(\Omega,\beta_+,\beta_-) \tag{3.9}$$

with $p_\phi = \pi\Pi_\phi$. The form of $V(\Omega,\beta_+,\beta_-)$ specifies each Bianchi model and is given in table 1. From (3.9), we derive the Hamiltonian equations:

$$\dot{\beta}_\pm = \frac{\partial H}{\partial p_\pm} = \frac{p_\pm}{H} \tag{3.10}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{12\phi^2 p_\phi}{(3+2\omega)H} \tag{3.11}$$

$$\dot{p}_+ = -\frac{\partial H}{\partial \beta_\pm} = -\frac{\partial V(\Omega,\beta_+,\beta_-)/\partial\beta_+}{2H} \tag{3.12}$$

$$\dot{p}_- = -\frac{\partial H}{\partial \beta_\pm} = -\frac{\partial V(\Omega,\beta_+,\beta_-)/\partial\beta_-}{2H} \tag{3.13}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = -12\frac{\phi p_\phi^2}{(3+2\omega)H} + 12\frac{\omega_\phi \phi^2 p_\phi^2}{(3+2\omega)^2 H} - 12\pi^2 R_0^6 \frac{e^{-6\Omega}U_\phi}{H} \tag{3.14}$$

$$\dot{H} = \frac{dH}{d\Omega} = \frac{\partial H}{\partial \Omega} = -72\pi^2 R_0^6 \frac{e^{-6\Omega}U}{H} + \frac{\partial V(\Omega,\beta_+,\beta_-)/\partial\Omega}{2H} \tag{3.15}$$

We set $N_i = 0$ and calculate that, whatever the Bianchi model, the lapse function is given by:

$$N = \frac{12\pi R_0^3 e^{-3\Omega}}{H} \tag{3.16}$$

Then, the relation between the time Ω and the proper time t is $dt = -Nd\Omega$.
Before starting the analysis of each Bianchi model, let us talk about some necessary conditions for isotropisation. By definition, the Universe isotropises if each metric function tends toward a common form, let us say R^2. From Misner parameterisation, we deduce that it implies $e^{-\Omega} \rightarrow R$ and $\beta_\pm \rightarrow 0$. A convenient measure of anisotropy is given by the a mesure of the root mean square anisotropy[21] $d\beta_+/dt^2 + d\beta_-/dt^2 = (d\beta_+/d\Omega^2 + d\beta_-/d\Omega^2)(d\Omega/dt)^2$ which have to decay such that isotropy occurs. Assuming that isotropisation is an asymptotic phenomenon arising when proper time diverges, it means that asymptotically $d\beta_\pm/dt \rightarrow 0$. It is this last necessary condition only that we will use in this work. It is not sufficient for isotropisation since it does not prevent $\beta_\pm$ to diverge or to tend toward a big constant but it is necessary if we want that the Hubble factors $H_i = \frac{dg_{ii}}{dt}g_{ii}^{-1}$ be asymptotically the same, i.e. tend toward the same value $d\Omega/dt$ in accordance with the fact that the Hubble constant is the same in any direction. Since the equations (3.10) and the expression for N lead to:

$$\frac{d\beta_\pm}{dt} = -\frac{p_\pm e^{3\Omega}}{12\pi R_0^3} \tag{3.17}$$

a stable isotropic state needs $p_\pm e^{3\Omega} \rightarrow 0$. We now look for the conditions allowing this limit.
Let us assume that isotropy leads to a static Universe, i.e. Ω tends toward a constant. Then $p_\pm$ must tend toward zero such that $p_\pm e^{3\Omega}$ vanishes. However, from (3.16) and (3.12-3.13), it comes that $dp_\pm/dt \propto \partial V/\partial\beta_\pm e^{3\Omega}$. Hence for the Bianchi I, VI_0 and $VIII$ models, when β_+ and Ω tend toward some constants, the conjugate momentum diverges with the proper time t since $\dot{p}_\pm$ tends toward a non vanishing constant and $p_+e^{3\Omega} \rightarrow \infty$. Thus from the reasonable assumption that isotropy happens when t diverges, we deduce that isotropisation can not lead to a static Universe for these three models (it would be deeply unnatural that a static Universe ends for a finite value of t). For the VII_0 and IX models, the demonstration is not so simple since then, when $\beta_\pm \rightarrow 0$ and $\Omega \rightarrow const$, $\partial V/\partial\beta_\pm \rightarrow 0$ and $\dot{p}_\pm$ vanishes. We will show below that for these models also, isotropy is not possible when Ω tends toward a constant. If now we assume that Ω diverges, nothing at this stage prevents the asymptotical vanishing of $p_\pm e^{3\Omega}$. Moreover, since $\beta_\pm$ tend toward some constants for a diverging value of Ω, we deduce that $d\beta_\pm/d\Omega \rightarrow 0$ otherwise $\beta_\pm$ would diverge with Ω. Consequently, isotropisation should arise when $\Omega \rightarrow \pm\infty$, $d\beta_\pm/d\Omega$ and $pe^{3\Omega} \rightarrow 0$. These conditions are independent each other and of the considered Bianchi class A models. We will have to check if each of them is respected for each presumed isotropic equilibrium state.

Let us compare them with isotropisation conditions defined by Collins and Hawking [108]. First, as in this last paper, we have assumed that isotropisation arises when $t \to \infty$. Second, in the next sections, for each Bianchi models, we will show that isotropisation needs $\Omega \to -\infty$. This is the first condition that defines isotropisation in Collins and Hawking's paper and which implies that Universe expands indefinitely. Third, the fact that $d\beta_\pm/dt$ and $d\beta_\pm/d\Omega$ tend toward zero satisfies their third condition meaning that "the anisotropy in the locally measured Hubble constant tends to zero". Thus from the necessary but not sufficient conditions stating that isotropisation needs $d\beta_\pm/dt \to 0$ and considering the field equations, we recover two of the four conditions of the Collins and Hawking definition for isotropy. It shows the physical meaning of the limit $p_\pm e^{3\Omega} \to 0$ regarding isotropisation. Moreover, assuming that $\beta_\pm$ tend toward some constants means that the matrix β whose components are the β_{ij} becomes a constant β_0 which is the fourth condition defining isotropy in [108].
In what follows, we study isotropic equilibrium states of each Bianchi class A model with curvature.

3.2.1 The Bianchi type II model

Field equations

To study the equilibrium points corresponding to asymptotic isotropic states, we will use the following variables, specific to the Bianchi type II model:

$$x_\pm = p_\pm H^{-1} \tag{3.18}$$

$$y = \pi R_0^3 \sqrt{U} e^{-3\Omega} H^{-1} \tag{3.19}$$

$$z = p_\phi \phi (3+2\omega)^{-1/2} H^{-1} \tag{3.20}$$

$$w = \pi R_0^2 e^{-2\Omega + 2(\beta_+ + \sqrt{3}\beta_-)} H^{-1} \tag{3.21}$$

They are independent since $x_\pm$, y, z and w are respectively functions of the independent quantities $p_\pm$, ϕ, p_ϕ and $\beta_\pm$. Then, the Hamiltonian (3.9) yields:

$$x_+^2 + x_-^2 + 24y^2 + 12z^2 + 12w^2 = 1 \tag{3.22}$$

We will consider this last expression as a constraint. It shows that the 5 variables $(x_\pm,y,z,w)$ are normalised. They allow us to rewrite the field equations as a first order equations system in the following way:

$$\dot{x}_+ = 72y^2 x_+ + 24w^2 x_+ - 24w^2 \tag{3.23}$$

$$\dot{x}_- = 72y^2 x_- + 24w^2 x_- - 24\sqrt{3}w^2 \tag{3.24}$$

$$\dot{y} = y(6\ell z + 72y^2 - 3 + 24w^2) \tag{3.25}$$

$$\dot{z} = y^2(72z - 12\ell) + 24w^2 z \tag{3.26}$$

$$\dot{w} = 2w(x_+ + \sqrt{3}x_- + 12w^2 + 36y^2 - 1) \tag{3.27}$$

with $\ell = \phi U_\phi U^{-1}(3+2\omega)^{-1/2}$. Since we want to keep ω and U undetermined, we will not explicit the form of $\ell(\phi)$. Then, the above system could not seem autonomous because $\ell = \ell(\phi)$ and thus it would be meaningless to look for its equilibrium point. However, it exists two possibilities such that it becomes autonomous. The first one is to consider ℓ as a function of (x,y,z,w) rather than ϕ. Such considerations are often used when one look for exact solutions of field equations: for instance, instead of considering the potential as a function of the scalar field, it can be easier to find exact solutions by assuming it depends on the metric functions[177]. However, in the general case, ℓ may not be written in this way in the whole range of ϕ. The second possibility, which applies whatever ℓ, is to consider an additional first order equation for ϕ which is derived from (3.11): $\dot{\phi}(z,\phi)$. Then, the system is autonomous. However, the constraint equation shows that scalar field equilibrium is not necessary for isotropisation contrary to other variables: $\dot{\phi}$ and ϕ can diverge whereas isotropy asymptotically occurs. Hence, we conclude that whatever the way we use such that the system be autonomous, equilibrium states are only determined by zeros of the system (3.23-3.27). The same reasoning may be applied for each Bianchi model. In what follows, we will use (3.23-3.27) to get the expressions for equilibrium points as some functions of $\ell(\phi)$ and then $\dot{\phi}(z,w)$ to get the asymptotical behaviour of the scalar field.

Isotropic equilibrium states

Equilibrium points

Several equilibrium points exist and we have to select those representing an isotropic equilibrium state. Immediately, we observe that the necessary conditions for isotropisation $d\beta_\pm/d\Omega \to 0$, imply that $x_\pm \to 0$ near equilibrium [1]. Since the equations (3.10) and the definitions of the variables $x_\pm$ will be identical for all the Bianchi models, these limits will be the same for each of them. Thus, we will systematically discard any equilibrium point with $x_\pm \neq 0$ or which is not defined by real values.
All the equilibrium points of the Bianchi type II model are summarised in the appendix. The first one is defined by $(y,w) = (0,0)$. Let us show that it is not consistent with isotropy. From equations (3.12-3.13), we derive that $p_+ - \sqrt{3}p_- = p_0$, p_0 being an integration constant. Then, using (3.10), it comes that $\dot\beta_+ - \sqrt{3}\dot\beta_- = p_0 H^{-1}$. Thus, isotropisation needs a diverging Hamiltonian but for the zero measure case with $p_0 = 0$. If now we consider (3.27) near $(y,w) = (0,0)$, we deduce that w behaves as $e^{-2\Omega}$ and vanishes when $\Omega \to +\infty$. Introducing this expression in (3.21), we derive that H should asymptotically behave as $e^{2(\beta_+ + \sqrt{3}\beta_-)}$ and thus should tend toward a constant value when Universe approaches isotropy. This contradict the fact that the Hamiltonian have to diverge and thus the set of points $(y,w) = (0,0)$ is not compatible with isotropisation.
The second set of points is not real and such that $x_\pm \neq 0$. Hence, it does not correspond to an isotropic equilibrium state and we discard it. The third set of points is such that $x_\pm = 0$ and respects the constraint. It writes $(x_+,x_-,y,z,w) = (0,0,\pm\sqrt{3-\ell^2}(6\sqrt{2})^{-1},\ell/6,0)$ and is real if $\ell^2 < 3$, implying that isotropisation is not possible if this last quantity diverges [2]. Indeed since asymptotically z behaves as $\ell/6$, it means that an isotropic stable state needs ℓ to tend to a constant value with no limit cycle otherwise $\dot z \not\to 0$ when $\Omega \to -\infty$. This is the only set of points representing an isotropic stable state and the only one we will consider below. The fourth set of points does not represent a stable isotropic state except if $\ell \to 1$. Then it tends toward the previous one. However, we will see below that the value $\ell = 1$ does not allow for isotropisation. Thus we discard it.

Monotonic functions

We rewrite the equation (3.15) with the normalised variables:

$$\dot H = -H(72y^2 + 24w^2) \tag{3.28}$$

Hence the Hamiltonian is a monotonic function of Ω with a constant sign. Then, from the lapse function expression (3.16), we deduce that Ω is a monotonic function of the proper time t. Therefore, if the Hamiltonian is initially positive (negative), $\Omega \to -\infty$ corresponds to late time(early time). We will not consider the case $\Omega \to +\infty$ since we will show below that it does not lead to isotropy. We conclude that late times isotropisation initially needs $H > 0$: this is the only necessary initial condition for this behaviour. Moreover, the Hamiltonian being of constant sign, it is the same for the variables y and w.

Asymptotic behaviours

We wish to evaluate the behaviours of some quantities in the neighbourhood of the equilibrium points $(x_+,x_-,y,z,w) = (0,0,\pm\sqrt{3-\ell^2}(6\sqrt{2})^{-1},\ell/6,0)$, i.e. when we approach isotropy in $\Omega \to -\infty$. Approximating (3.27) near equilibrium by $w(1-\ell^2)$, we find that asymptotically w behaves as $e^{(1-\ell^2)\Omega}$.

From this last expression and approximating (3.23) by $(3-\ell^2)x_+ - 24w^2$, we deduce that x_+ behaves as the sum of two terms $e^{2(1-\ell^2)\Omega}$ and $e^{(3-\ell^2)\Omega}$. Since isotropy needs $x_+ \to 0$ and $\ell^2 < 3$, we derive it only occurs if $\Omega \to -\infty$ and $\ell^2 < 1$. These two limits are in accordance with the vanishing of $x_\pm$ and w which is necessary to reach the equilibrium isotropic state. Consequently $x_\pm$ asymptotically behave as $e^{2(1-\ell^2)\Omega}$. Let us note that the two limits $x_\pm \to 0$ and $\Omega \to -\infty$, necessary for isotropisation, are compatible and justify our assumption that it takes place for a diverging value of t: when $\Omega \to -\infty$ and the Universe isotropises, it is expanding and there is no physical meaning to consider that it ends for a finite value of the proper time. Let us show that the value $\ell^2 = 1$ has to be discarded. If $1-\ell^2 \to 0$ faster than Ω^{-1}, w tends toward a non vanishing constant and is not compatible with isotropy. If $1-\ell^2 \to 0$ slower than Ω^{-1}, from (3.15) we deduce that near equilibrium $H \to e^{-2\Omega}$. Then from (3.12), it comes that $p_+ \to e^{-2\Omega}$ and it follows

1. Note that we have then, near equilibrium, $\dot x_\pm \to 0$ and $x_\pm \to 0$. This is a consequence of the field equations and values of the equilibrium points near isotropy. It means that $x_\pm$ should be integrable in the Lebesgue sense in the neighbourhood of equilibrium. We will see that it is actually the case when we will calculate the asymptotical behaviours of $x_\pm$.

2. We will not take into account the case for which ℓ^2 would have a chaotic behaviour such that it stays smaller than 3.

from (3.18) that x_+ tends toward a non vanishing constant. Hence, the limit $\ell^2 \to 1$ is not compatible with isotropy. The above reasoning concerning the case for which $1-\ell^2 \to 0$ faster than Ω^{-1} will stay valid for any Bianchi model. However, when $1-\ell^2 \to 0$ slower than Ω^{-1}, it will be valid only for Bianchi type II, VI_0 and $VIII$ models. For Bianchi type VII_0 and IX models, the situation is different because when $\beta_\pm \to 0$ during isotropisation, $p_\pm$ varies slower than $e^{-2\Omega}$ since $\partial V/\partial\beta_\pm e^{4\Omega} \to 0$. However, we will show below that near equilibrium, the value $\ell = 1$ can also be excluded.
Now it is possible to show that $p_\pm e^{3\Omega}$ vanishes when $\Omega \to -\infty$. Writing $\dot{p}_\pm/H$ as a function of $x_\pm$ and w and using their asymptotical behaviours, we calculate that $\dot{p}_\pm/p_\pm$ tends toward the constant $-(1+\ell^2)$. Consequently $p_\pm e^{3\Omega} \to e^{(2-\ell^2)\Omega} \to 0$ when $\Omega \to -\infty$ and the necessary but not sufficient condition for isotropisation is respected. Moreover, from (3.28), we calculate that asymptotically $\dot{H}H^{-1}$ tends toward ℓ^2-3. Thus $H \to e^{(\ell^2-3)\Omega}$ and diverges since $\ell^2 < 1$: therefore, although determined independently, the asymptotical behaviours of $p_\pm$ and H agreed with these of $x_\pm = p_\pm H^{-1}$.

Concerning the scalar field, we can find a differential equation whose solution asymptotically behaves in the same way as ϕ when $\Omega \to -\infty$ by expressing equation (3.11) with the normalised variables and considering its asymptotical limit near equilibrium. It comes:

$$\dot{\phi} = \frac{2\phi^2 U_\phi}{(3+2\omega)U} \tag{3.29}$$

This important equation allows us to get the asymptotical behaviour of ϕ near equilibrium and consequently that of ℓ.

From the asymptotical behaviour of H and the expression (3.16) for the lapse function, it is possible to get the isotropic part of the metric, $e^{-\Omega}$, as a function of the proper time. If ℓ^2 tends toward a non vanishing constant, then $e^{-\Omega} \to t^{\ell^{-2}}$. If ℓ^2 tends to 0 faster than $(-\Omega)^{-1}$, $e^{-\Omega}$ behaves like an exponential. Let us show that it is always the case. Equation (3.29) can be rewritten as $d\phi/d\Omega = 2\ell^2 U(U_\phi)^{-1}$ and then $U \propto e^{2\int \ell^2 d\Omega}$ when $\Omega \to -\infty$. Introducing this expression and the expression for H into (3.19), we get $y \propto e^{-\ell^2\Omega + \int \ell^2 d\Omega}$. Thus, if ℓ^2 vanishes slower than $(-\Omega)^{-1}$, y diverges or tends to 0 instead of a non vanishing constant and there is no equilibrium.

Hence, when an isotropic equilibrium state is reached with $\ell \to 0$, ℓ^2 always vanishes faster than $(-\Omega)^{-1}$ and Universe always tends toward a De Sitter one. This proof for $\ell = 0$ relies on the asymptotic form of ϕ, H and the definition of y. It will be valid for all the Bianchi models since we will see that these 3 quantities keep the same forms each time the same set of equilibrium points is considered.
From (3.19) and the asymptotical forms for $\Omega(t)$, we deduce that the potential behaves as t^{-2} when ℓ tends toward a non vanishing constant, or as a non vanishing constant otherwise (the same behaviour held when we consider the Bianchi type I model[105]). Concerning the 3-curvature which can be expressed as $R^{(3)} = e^{2\Omega+4(\beta_+ + \sqrt{3}\beta_-)}$, it is obvious that near isotropy, it tends to zero, showing that the Universe becomes spatially flat.
Let us note the importance of the potential for isotropisation of the Bianchi type II model. If we consider $U = 0$, we get from the field equations (3.12) and (3.15) that $H = p_+ + p_0$ and thus $\dot{\beta}_+ = 1 - p_0 H^{-1}$, p_0 being an integration constant. It follows that $\dot{\beta}_+$ does not asymptotically vanish and isotropy can not be reached.

Partial equilibrium[3]

Above we have defined the isotropic equilibrium states such that all the variables reach equilibrium. However, this last statement is not mandatory: all of them have not to reach equilibrium such that $x_\pm \to 0$ in a stable way. In this case, as $\Omega \to -\infty$, the concerned variables would stay bounded but their derivatives would not asymptotically vanish: they should oscillate indefinitely (around a constant or not) and thus, their derivatives should also oscillate around zero without beeing damped. What are the variables able to behave in this way?

We can reasonably assume that it is not $x_\pm$, otherwise it would mean that the variation of the Hubble constant would be anisotropic. Since $x_\pm$ and $\dot{x}_\pm$ have to vanish, it implies that w also asymptotically vanishes and consequently, $\dot{w} \to 0$. Finally the only variables whose equilibrium is not necessary to isotropy and whose derivatives could be oscillating are y and z. Under these assumptions, what about ℓ behaviour?
Equation (3.26) writes asymptotically $\dot{z} = y^2(72z - 12\ell)$. Since we have assumed that $\dot{z}$ was oscillating, it shows that ℓ can not tend to a constant, diverge monotonically or diverge oscillatory if the oscillations are

3. I thank one of the referees criticisms from which this section is inspired.

not large enought($\ell = \Omega^{1/3} + \cos\Omega$ as instance) otherwise the sign of $\dot{z}$ would be constant. Consequently, only oscillatory ℓ with sufficiently large amplitudes and not tending to a constant may allow oscillations of z too ($\ell = \Omega \sin\Omega$ or $\ell = n\cos\Omega$ with n a constant larger than the largest amplitude of $6z$ as instance). In this case, the results of the previous sections do not apply since all the variables do not reach equilibrium but an isotropic equilibrium state eventually occurs with $(x_\pm, w)$ only reaching equilibrium. Then, since ℓ can be a regular functions as well as having a chaotic behaviour, it is not possible to give more characteristics about it or the way $x_\pm$ would reach equilibrium. Hence, the main result of this subsection is mainly a limitation of the previous subsections results which will be valid for all the Bianchi class A models.

Let us examine the folowing example for the Bianchi type II model:

Since $x_\pm$ and w vanish, asymptotically the constraint is $24y^2 + 12z^2 = 1$. In this limit, considering that (3.26) is essentially equivalent to the constraint equation under (3.25), (3.25) is the only nontrivial equation. In this asymptotic reduction, (3.25) can be regarded as the equation for ℓ in terms of y, which can be written as:

$$\ell = \frac{\dot{v}}{2+v}\frac{1}{\sqrt{1-v}} + \sqrt{1-v},$$

where $v = 2(36y^2 - 1)$ and $6z = \sqrt{1-v}$. For any function for $v(\Omega)$, the equation

$$\dot{\phi} = \frac{12\phi}{\sqrt{3+2\omega}} z,$$

determines ϕ as a function of Ω for a given ω. Then, the definition of ℓ,

$$\ell = \frac{\phi}{\sqrt{3+2\omega}}\frac{U_\phi}{U},$$

determines U as a function of ϕ, provided that $\phi - \Omega$ relation is invertible. Here, if v is bounded by a positive constant from below and if $1 - v$ is non-negative, one can easily check that $x_\pm$ and w have required asymptotic behavior. Thus, the only possible constraint on $v(\Omega)$ is the invertibility of the $\phi - \Omega$ relation. For example, the choice:

$$v = v_0 + \frac{1}{\Omega}\sin^2\Omega^3$$

satisfies this condition, if v_0 is a constant in the range $0 < v_0 < 1$. However, for this choice, ℓ, $\dot{y}$ and $\dot{z}$ diverge oscillatorily as $\Omega \to -\infty$, although the isotropization condition $x_\pm \to 0$ as $\Omega \to -\infty$ is satisfied and (y, z) are bounded.

3.2.2 The Bianchi type VI_0 and VII_0 models

The results are similar to those of Bianchi type II model.

Field equations

We will use the following variables for both Bianchi type VI_0 and VII_0 models:

$$x_\pm = p_\pm H^{-1} \tag{3.30}$$
$$y = \pi R_0^3 e^{-3\Omega} U^{1/2} H^{-1} \tag{3.31}$$
$$z = p_\phi \phi (3+2\Omega)^{-1/2} H^{-1} \tag{3.32}$$
$$w_\pm = \pi R_0^2 e^{-2\Omega + 2\beta_+ \pm 2\sqrt{3}\beta_-} H^{-1} \tag{3.33}$$

They are independent since they respectively depend on $p_\pm$, ϕ, p_ϕ and a combining of $\beta_\pm$. The definitions of $x_\pm$, y and z are the same as these of the Bianchi type II model. The Hamiltonian is written:

$$x_+^2 + x_-^2 + 24y^2 + 12z^2 + 12(w_+ \pm w_-)^2 = 1 \tag{3.34}$$

and the field equations become:

$$\dot{x}_+ = 72y^2 x_+ + 24(x_+ - 1)(w_- \pm w_+)^2 \tag{3.35}$$
$$\dot{x}_- = 72y^2 x_- + 24x_-(w_- \pm w_+)^2 + 24\sqrt{3}(w_-^2 - w_+^2) \tag{3.36}$$
$$\dot{y} = y(6\ell z + 72y^2 - 3 + 24(w_- \pm w_+)^2) \tag{3.37}$$

$$\dot{z} = y^2(72z - 12\ell) + 24z(w_- \pm w_+)^2 \tag{3.38}$$
$$\dot{w}_+ = 2w_+\left[x_+ + \sqrt{3}x_- + 12(w_- \pm w_+)^2 + 36y^2 - 1\right] \tag{3.39}$$
$$\dot{w}_- = 2w_-\left[x_+ - \sqrt{3}x_- + 12(w_- \pm w_+)^2 + 36y^2 - 1\right] \tag{3.40}$$

The $\pm$ symbols in equations (3.34-3.40) correspond respectively to the Bianchi type VI_0 and VII_0 models. For the first model, the constraint shows that the variables are normalised since w_+ and w_- are positive. For the second one, because of the minus sign, both w_+ and w_- could diverge. Then the constraint will be respected only if the sum $w_+ - w_-$ tends toward a constant. We will show below that isotropy is only compatible with finite values of $w_\pm$. Consequently, the isotropic states we are looking for are reached for some bounded values of the variables whatever the Bianchi type VI_0 or VII_0 models. Now, we can also show that isotropisation of Bianchi type VI_0 and VII_0 models may not arise for a finite value of Ω. Indeed, if Ω tends toward a constant when the proper time diverges, $d\Omega/dt$ vanishes and thus, from (3.16), it comes that H vanishes. But then, $w_\pm$ should diverge which is not compatible with the equilibrium as shown above. Hence, for the Bianchi type VI_0 and VII_0 models, isotropisation does not lead to a static Universe.

Isotropic equilibrium states

Equilibrium points

All the equilibrium points with finite values of w_+ and w_- are referenced in the appendix. Following the same reasoning as for Bianchi type II model, the only set of equilibrium points compatible with an isotropic stable state is $(x_+,x_-,y,z,w_+,w_-) = (0,0,\pm\sqrt{3-\ell^2}(6\sqrt{2})^{-1},\ell/6,0,0)$, implying that $\ell^2 < 3$. It is equivalent to the points found for Bianchi type II model and, for the same reasons, ℓ have to tend to a constant with no limit cycle. Another interesting set of points is given by $(x_+,x_-,y,z,w_\pm,w_\pm) = ((\ell^2-1)(\ell^2+8)^{-1},\pm\sqrt{3}(\ell^2-1)(\ell^2+8)^{-1},\pm\sqrt{12-3\ell^2}\left[2(\ell^2+8)\right]^{-1},3\ell(2\ell^2+16)^{-1},0,\pm\sqrt{-\ell^4+5\ell^2-4}\left[2(\ell^2+8)\right]^{-1})$. However, in this case $x_\pm \to 0$ only if $\ell^2 \to 1$ and we recover the values of the previous set of points for this particular limit of ℓ^2 which does not allow isotropisation as it will be shown below. Other sets exist but are complex valued and can thus be discarded. At last, as for Bianchi type II model, let us show that the set of equilibrium points such that $(y,w_+,w_-) = (0,0,0)$ implies that $x_\pm$ do not vanish.
In this case we deduce from (3.39-3.40) that $w_\pm$ behave as $e^{-2\Omega}$ and thus isotropy would arise when $\Omega \to +\infty$. Then, using the definition (3.33) for $w_\pm$, we derive that H should be a constant near isotropy and, considering equations (3.12-3.13), we find that asymptotically $\dot{p}_\pm$ should vary as $p_{0\pm}e^{-4\Omega}$ ($p_{0\pm}$ being some constants) for Bianchi type VI_0 or even slower for the VII_0 type since $\beta_\pm$ might tend toward some vanishing constants. It would follow that $p_\pm \to p_{1\pm}$, $p_{1\pm}$ being some integration constants. However, in this case $\dot{\beta}_\pm$ would tend asymptotically toward the constants $p_{1\pm}H^{-1}$ and isotropisation could not occur for a diverging value of Ω. Hence, $(y,w_+,w_-) = (0,0,0)$ is not compatible with isotropisation except for the special case of zero measure $p_{1\pm} = 0$.

Monotonic functions

What about monotonic functions? We can rewrite equation (3.15) as follows:

$$\dot{H} = -H\left[72y^2 + 24(w_+ \pm w_-)^2\right] \tag{3.41}$$

As for the Bianchi type II model, (3.41) shows that the Hamiltonian is a monotonic function of constant sign which determines if isotropisation occurs at early or late times depending if the Hamiltonian is initially negative or positive. Moreover, it follows that y and $w_\pm$ are of constant sign.

Asymptotic behaviours

Making the same approximation as for subsection 3.2.1, we find that near an isotropic equilibrium state, $w_\pm \to e^{(1-\ell^2)\Omega}$. Then, assuming that isotropisation arises for $\Omega \to -\infty$ as we will show it below, it follows from (3.41) that $H \to e^{-2\Omega}$ when $\ell \to 1$ and then, near equilibrium, $w_\pm$ should tend toward some non vanishing constants. Thus the value $\ell = 1$ does not agree with isotropisation.
Concerning $x_\pm$, from (3.35-3.36), we deduce that they asymptotically behave as the sum of two exponentials $e^{2(1-\ell^2)\Omega}$ and $e^{(3-\ell^2)\Omega}$, showing again that these quantities will tend toward zero only if $\Omega \to -\infty$ and $\ell^2 < 1$. These two limits allow vanishing of $w_\pm$, which is necessary to reach the equilibrium states.
All these elements show that near isotropy $w_\pm$ is bounded. Indeed, for (x,y,z) to reach equilibrium, one

only needs that $w_+ - w_- \rightarrow 0$ whatever the particular asymptotical behaviours of $w_\pm$. Then only using this last limit, we have recovered the asymptotical behaviours for $x_\pm$ which implies that $\ell^2 < 1$. From (3.39-3.40) and these last expressions and condition, it is then possible to get the particular behaviours of $w_\pm$, which show that, near isotropy, these variables always vanish and are then bounded.
Again, we are able to calculate that $p_\pm e^{3\Omega}$ tends to 0 as $e^{(2-\ell^2)\Omega}$, thus filling the necessary but not sufficient conditions for isotropisation we defined above. From equation (3.41) we derive that asymptotically H behaves as $e^{(\ell^2-3)\Omega}$. The form of equation (3.11) for $\dot\phi$ being unchanged whatever the Bianchi model as well as those of ℓ and z near equilibrium, we recover the same differential equation as (3.29) giving the asymptotical behaviour for the scalar field.
Since H and $\dot\phi$ when $\Omega \rightarrow -\infty$, and N and y have the same forms as for the Bianchi type II model, we find the same behaviours for $e^{-\Omega}$ and U as a function of the proper time depending if ℓ tends or not toward a vanishing constant. Again, the 3-curvature ${}^3R$ tends to zero since $\beta_\pm$ become constant when Ω diverges negatively.

3.2.3 The Bianchi type $VIII$ and IX models

Field equations

We will use the following variables:

$$x_\pm = p_\pm H^{-1}$$
$$y = \pi R_0^3 e^{-3\Omega} U^{1/2} H^{-1}$$
$$z = p_\phi \phi (3+2\Omega)^{-1/2} H^{-1}$$
$$w_p = \pi R_0^2 e^{-2\Omega+2\beta_+} H^{-1}$$
$$w_m = \pi R_0^2 e^{-2\Omega-2\beta_+} H^{-1}$$
$$w_- = e^{2\sqrt{3}\beta_-}$$

The variables $x_\pm$, y and z are the same as those defined for the Bianchi type II model. w_p and w_m are not independent because both are related to β_+. Near isotropy, we will have $w_m \propto w_p \propto e^{-2\Omega}H^{-1}$. w_- is positive. The constraint equation is written:

$$x_+^2 + x_-^2 + 24y^2 + 12z^2 + 12[w_p^3(1+w_-^4) \pm 2w_-(w_m w_p)^{3/2}(1+w_-^2) + w_-^2(w_m^3 - 2w_p^3)](w_-^2 w_p)^{-1} = 1$$

and for the field equations it comes:

$$\dot x_+ = 72y^2 x_+ + 24\{w_p^3(x_+-1)(1+w_-^4) \pm w_-(1+2x_+)(w_m w_p)^{3/2}(1+w_-^2) + w_-^2\left[(2+x_+)w_m^3 - 2(x_+-1)w_p^3\right]\}(w_-^2 w_p)^{-1} \quad (3.42)$$

$$\dot x_- = 72y^2 x_- + 24\{w_p^3\left[w_-^4(x_- - \sqrt{3}) + x_- + \sqrt{3})\right] \pm w_-(w_m w_p)^{3/2}[w_-^2(-\sqrt{3}+2x_-) + (\sqrt{3}+2x_-)] + w_-^2 x_-(w_m^3 - 2w_p^3)\}(w_-^2 w_p)^{-1} \quad (3.43)$$

$$\dot y = y\{6\ell z + 72y^2 - 3 + 24[w_p^3(1+w_-^4) \pm 2(w_m w_p)^{3/2} w_-(1+w_-^2) + w_-^2(w_m^3 - 2w_p^3)](w_-^2 w_p)^{-1}\} \quad (3.44)$$

$$\dot z = y^2(72z - 12\ell) + 24z[w_p^3(1+w_-^4) \pm 2(w_m w_p)^{3/2} w_-(1+w_-^2) + w_-^2(w_m^3 - 2w_p^3)](w_-^2 w_p)^{-1} \quad (3.45)$$

$$\dot w_p = w_p\{-2 + 2x_+ + 72y^2 + 24[w_p^3(1+w_-^4) \pm 2w_-(w_m w_p)^{3/2}(1+w_-^2) + w_-^2(w_m^3 - 2w_p^3)](w_-^2 w_p)^{-1}\} \quad (3.46)$$

$$\dot w_m = w_m\{-2 - 2x_+ + 72y^2 + 24[w_p^3(1+w_-^4) \pm 2w_-(w_m w_p)^{3/2}(1+w_-^2) + w_-^2(w_m^3 - 2w_p^3)](w_-^2 w_p)^{-1}\} \quad (3.47)$$

$$\dot w_- = 2\sqrt{3} w_- x_- \quad (3.48)$$

$\pm$ being related respectively to the Bianchi type $VIII$ and IX models. For sake of completeness, we have written differential equations for each variable w_p and w_m. However, they are equivalent. The constraint

shows that the variables are not necessarily normalised: if one of them diverges near isotropy, it have to be counterbalanced by the divergence of w_p and w_m. Thus, if we show that isotropy does not arise for unbounded values of w_p and w_m, it will mean that it only happens for some finite values of the variables.
To reach this goal, we will write that $w_p \to w_m \to w$ and $w_- \to 1$. This is justified because isotropy needs $\beta_\pm \to 0$ and we will see below that an isotropic equilibrium state effectively implies $w_- \to 1$. In this case, for Bianchi type $VIII$ model, all the variables in the constraint are positives and thus bounded. Concerning the Bianchi type IX model, let us assume that w diverges. Then, putting $x_\pm = 0$, from the constraint we have asymptotically $3w^2 \to 2y^2 + z^2 - 1/12$ and from the equation for $\dot{w}$, $3w^2 \to 3y^2 - 1/12$, implying that asymptotically $z^2 \to y^2$ and diverges as w^2. However, with these limits we get from the equations for $\dot{y}$ and $\dot{z}$ that $\dot{y} \to 6\ell z^2 - 3z$ and $\dot{z} \to -12\ell z^2 + 2z$. Then equilibrium for y and z can only be reached if $z \to 0$ which is in disagreement with the assumption on the divergence of w. Hence, an isotropic equilibrium state is not possible if w_p and w_m diverge. It follows, in the same way as for the Bianchi type VI_0 and VII_0 models, that isotropisation of Bianchi type $VIII$ and IX models for a finite value of Ω is impossible.
We can also show that w_p and w_m may not tend toward some non vanishing constants. Let us assume that it is actually the case and consider 2 constants w and α such that $w_p \to w$ and $w_m \to \alpha w$. We introduce these limits in equations (3.42-3.43) with $x_\pm = 0$ and get respectively:

$$\dot{x}_+ = -24w^2(1 + w_-\alpha^{3/2}(1 + w_-^2) - 2w_-^2(1 + \alpha^3) + w_-^4)w_-^{-2} \tag{3.49}$$

$$\dot{x}_- = -24\sqrt{3}w^2(w_-^2 - 1)(1 - \alpha^{3/2}w_- + w_-^2)w_-^{-2} \tag{3.50}$$

Then, for Bianchi type $VIII$ model, we derive that equilibrium for $x_\pm$ will be reached only if α tends toward a complex value $(-1)^{2/3}$ or/and w_- is negative, which is impossible. For the Bianchi type IX model, equilibrium for $x_\pm$ may be reached if $w_p \to w_m$ (i.e. $\beta_\pm \to 0$) and $w_- \to 1$. Then, looking for the equilibrium points, the only ones which may be real and such that w_p and w_m be different from 0 are $(x_+,x_-,y,z,w_p,w_m,w_-) = (0,0,\pm(6\ell)^{-1},(6\ell)^{-1},\pm(1-\ell^2)^{1/2}(6\ell)^{-1})$. They check the constraint equation and are real if $\ell^2 < 1$. Then we calculate that w_p and w_m behave like $\pm\left[(1-\ell^2)(1-e^{\frac{4\Omega(\ell^2-1)+\omega_0}{\ell^2}} + 36\ell^2)^{-1}\right]^{1/2}$ and thus reach equilibrium when $\Omega \to +\infty$. Meantime, starting from this last expression and introducing it in the equation for x_+, it comes that x_+ tends toward a complex value when $\Omega \to +\infty$ and thus these equilibrium points are excluded. Numerical simulations seem to confirm the absence of equilibrium for these values of $(x_+,x_-,y,z,w_p,w_m,w_-)$.

Isotropic equilibrium states

Equilibrium points

To find the equilibrium points we have to consider the equations (3.42-3.45), (3.48) and one of the equations (3.46) or (3.47) since both w_m and w_p depend on β_+. However the solutions of the equations system thus defined can not be easily calculated. Consequently, we will only take into account the solutions such that $(x_\pm,w_p,w_m) = (0,0,0)$ and which are compatible with isotropy. Then the solutions reduce to the set of equilibrium points $(x_+,x_-,y,z,w_p,w_m,w_-) = (0,0,\pm\sqrt{3-\ell^2}(6\sqrt{2})^{-1},\ell/6,0,0,1)$ which will be real if $\ell^2 < 3$ and respect the constraint equation. It is equivalent to the sets found for the previous models and once again, ℓ have to tend to a constant with no limit cycle such that equilibrium be reached. Note that it is such that $\beta_- \to 0$ since $w_- \to 1$.

Monotonic functions

We can rewrite (3.15) in the following form:

$$\dot{H} = -H[72y^2 + 24(\pm 2\frac{w_p^{1/2}w_m^{3/2}}{w_-} \pm 2w_p^{1/2}w_m^{3/2}w_- - 2w_p^2 + \frac{w_p^2}{w_-^2} +$$

$$w_p^2 w_-^2 + \frac{w_m^3}{w_p})] \tag{3.51}$$

$$\tag{3.52}$$

We immediately see that it is not a monotonic function and that its sign is indefinite. Thus Ω is not a monotonic function of t and it is not possible to determine if isotropisation, corresponding to $\Omega \to -\infty$, arises at early or late proper times.

Asymptotic behaviours

Near equilibrium, it is possible to approximate equation (3.46) by $w_p(1-\ell^2)$ implying that w_p tends toward $e^{(1-\ell^2)\Omega}$. The same conclusion arises for w_m. In the same way as the previous subsection, one can show that the value $\ell^2 = 1$ is not agreed with isotropisation. Introducing asymptotical expressions for w_p and w_m in the equations (3.42-3.43), we find that $x_\pm$ behave as the sum of two exponentials, $e^{2(1-\ell^2)\Omega}$ and $e^{(3-\ell^2)\Omega}$. Thus, once again, isotropisation needs $\Omega \to -\infty$ and $\ell^2 < 1$ implying that $x_\pm$ behave as $e^{2(1-\ell^2)\Omega}$. As for Bianchi type II model, it is possible to show that $p_\pm e^{3\Omega} \to e^{(2-\ell^2)\Omega}$ and thus vanish. Moreover H behaves again as $e^{(\ell^2-3)\Omega}$. The Hamiltonian equation (3.11) for $\dot\phi$ being independent of considered Bianchi model and the definition and asymptotical value of z being the same as for Bianchi type II model, we find the same differential equation (3.29), giving asymptotically the behaviour of the scalar field. Anew, since H and $\dot\phi$ when $\Omega \to -\infty$, and N and y have the same forms as for Bianchi type II model, the discussion about the forms of the metric functions near equilibrium is the same and they behave as power or exponential law of the proper time depending on the asymptotical value of ℓ.

3.3 Discussion

We have found some necessary conditions for isotropisation of Bianchi class A models with curvature for a minimally coupled scalar tensor theory. We have seen that the Universe has to expand ($\Omega \to -\infty$), justifying the assumption that t should be diverging, and that the ratio between the conjugate momentum and the Hamiltonian should vanish. Our results do not concern the class of theories for which ℓ prevents the equilibrium of z and y. As shown in subsection 3.2.1, such ℓ should be oscillating with significant amplitude and not tending to a constant. Hence, they concern the ℓ which tend to a constant, diverge monotonically or even with negligible oscillations. In these cases, our main result is:

A necessary condition for isotropisation of Bianchi class A models with curvature for General Relativity plus a massive scalar field, whatever the considered Brans-Dicke coupling function and potential, will be that $\phi U_\phi U^{-1}(3+2\omega)^{-1/2}$ tends to a constant ℓ such that $\ell^2 < 1$. For Bianchi type II, VI_0 and VII_0 models, it arises at late times if the Hamiltonian is positive, at early times otherwise. For Bianchi type $VIII$ and IX models, the time of isotropisation is undetermined. If isotropisation arises with $\ell \neq 0$ the metric functions tend toward a power inflationary law $t^{\ell^{-2}}$ and the potential vanishes as t^{-2}. If it arises as $\ell = 0$, the Universe tends toward a De Sitter model and the potential to a constant. In any case, isotropisation requires late time accelerated expansion and the Universe becomes spatially flat.

Necessary condition for isotropisation determines an asymptotical limit that the scalar field have to respect. It can be compared to the limit required such that scalar tensor theories be compatible with solar system tests when $U = 0$, i.e. $\omega \to \infty$ and $\omega_\phi \omega^3 \to 0$. To evaluate ℓ, we need to know the asymptotical behaviour of the scalar field. It comes:

The value of the scalar field when the Universe reaches an equilibrium isotropic state with an asymptotically constant ℓ is the value of the function ϕ defined by $\dot\phi = 2\phi^2 U_\phi (3+2\omega)^{-1} U^{-1}$ when $\Omega \to -\infty$.

Although Bianchi type IX model contains the closed FLRW solutions, when the Universe isotropises and we consider a minimally coupled and massive scalar field, it is infinitely expanding. Moreover, the common late time attractor of all the isotropising solutions is not oscillating. This fact may seem astonishing for the Bianchi type $VIII$ and IX models. However, in [128], it has been observed that despite the mixmaster behaviour of Bianchi type $VIII$ model at early time, its late time behaviour can be non oscillating. If we compare the results got when no curvature is present [105] with those of this paper, few differences appear. The asymptotical behaviours of the scalar field and isotropic part of the metric are the same in both papers, partly because the Hamiltonian and lapse function asymptotically behave in the same way. However the variations of the functions describing the anisotropy, $\beta_\pm$, are different since the conjugate momenta are not constant in presence of curvature. The fundamental difference comes from the interval of ℓ allowing for isotropy. For the Bianchi type I model, it was $\ell^2 < 3$ and decelerated dynamics was possible. For the models with curvature, we have $\ell^2 < 1$, implying that isotropisation requires late time accelerated expansion. This is due to the presence of curvature which reduces the interval of values of ℓ related to the Bianchi type I model. Hence, late time accelerated expansion finds a natural explanation through the fact that our Universe is isotropic. Other problems are naturally solved by isotropisation: asymptotically, the 3-curvature vanishes thus solving the flatness problem. It comes from the fact that during isotropisation $\beta_\pm$ tend toward a constant whereas $\Omega \to -\infty$. In the same way, the small value of the cosmological constant could be explained by the fact that when Universe isotropises and ℓ does not vanish, the potential, which mimics a dynamical cosmological constant vanishes. If ℓ vanishes, the potential tends to a non vanishing constant but it is not necessary small except if we fine tune it.

To complete this study, let us write few words about Bianchi class B models. Their Hamiltonian formulation is different from those of class A and has been studied in [178]. It needs to redefine the divergence theorem in a non-coordinated basis. Then the Bianchi type V Hamiltonian writes as the one of Bianchi type I with an additional constraint $p_+ = 0$. Consequently for Bianchi type V model which contains the solutions of open FLRW model, isotropisation follows the same rules as those of Bianchi type I model described in [105]. The nature of the other class B Hamiltonians is totally different and will not be considered here.

Let us examine some results usually considered in the literature.

The "No Hair theorem" of Wald[49] states that General Relativity with a scalar field and a cosmological constant isotropises toward a De Sitter Universe. Here, as for the Bianchi type I model, when $\ell \to 0$ and if the minimally coupled scalar tensor theory isotropises [4], it will tend toward a De Sitter Universe. This generalise the "No Hair theorem" which takes into account only the case $U = cte$ for which $\ell = 0$.

It is really shocking that it exists only one set of equilibrium points shared by all the Bianchi models and representing the only possible isotropic stable equilibrium state. However, despite a careful analysis we have not found any additional points with such properties. One way to check if this statement is true is to select some special forms of U and ω and then to verify if the conditions for isotropisation of the theory thus defined and the asymptotical value of $e^{-\Omega}$ are in agreement with our results. It can be easily done with the theory defined by an exponential potential $U = e^{k\phi}$ and a Brans-Dicke coupling function $\sqrt{3+2\omega}\phi^{-1} = \sqrt{2}$ whose isotropisation has been extensively studied in the literature using different methods. In this case $\ell^2 = k^2/2$. We have collected the conclusions of different papers and have compared them with ours. In [86, 85], it is shown that isotropisation arises at late time when $k^2 < 2$ (except the contracting Bianchi type IX models) and lead to a De Sitter Universe when $k = 0$ or to a power law of the form $t^{2k^{-2}}$ for the metric functions otherwise. If $k^2 > 2$, the Bianchi type I, V, VII and IX models might isotropise at late times. Concerning the Bianchi type I model, we have shown in [105] that a necessary condition for isotropisation will be $k^2 < 6$ but it was impossible for larger values. For the models of class A with curvature, from the present paper we deduce that isotropisation is possible only when $k^2 < 2$ and always comes with late time accelerated expansion. The asymptotic behaviour of the metric functions is in accordance with that of [86, 85]. A difference is that Bianchi type VII_0 and IX should not isotropise if $k^2 > 2$. Concerning the Bianchi type VI_0 model, our results agree with these of [96]. For the Bianchi type V model, they are the same as these of the Bianchi type I model in accordance with [86]. Hence, concerning the special case of an exponential potential, there are few differences between our results and those of others papers. It seems to confirm the presence of a unique set of equilibrium points shared by all the Bianchi class A models and representing an isotropic equilibrium state. Of course, the case of an exponential potential could be a particular one and thus other types of potentials should be studied to check the results of the present paper. Note that, isotropic state is not the only possible late time equilibrium state. As written above or shown in the appendix, other ones exist, for instance with $x_\pm \neq 0$, but they do not correspond to an isotropic Universe.

To conclude, Universe isotropisation requires late time accelerated expansion because of the curvature. Then, it becomes flat and the potential vanishes as t^{-2} or tends toward a constant. These features fit well with the current observations and leave the door open to geometrical and physical generalisations of standard cosmological framework. In a next work, we will take into account the presence of a perfect fluid.

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4. Do not forget that ℓ tending toward a constant is a necessary but not sufficient condition for isotropisation.

3.4 Appendix

In this appendix, we present all the equilibrium points of Bianchi type II, VI_0 and VII_0 models.

Bianchi type II model :

- $(y,w) = (0,0)$
- $(x_+,x_1,y,z,w) = (1,\sqrt{3},0,0,\pm i/2)$
- $(x_+,x_1,y,z,w) = (0,0,\frac{\pm\sqrt{3-\ell^2}}{(6\sqrt{2})},\ell/6,0)$
- $(x_+,x_1,y,z,w) = (\frac{\ell^2-1}{\ell^2+8},\sqrt{3}\frac{\ell^2-1}{\ell^2+8},\pm\frac{\sqrt{12-3\ell^2}}{2(\ell^2+8)},\frac{3\ell}{2(\ell^2+8)},\pm\frac{\sqrt{(\ell^2-1)(\ell^2-4)}}{2(\ell^2+8)})$

Bianchi type VI_0 and VII_0 models :

- $(y,w_+,w_-) = (0,0,0)$
- $(x_+,x_-,y,w_+,w_-) = (1,0,0,w_+,w_+)$
- $(x_+,x_-,y,z,w_+,w_-) = (1,-\sqrt{3},0,0,0,\pm i/2)$
- $(x_+,x_-,y,z,w_+,w_-) = (1,\sqrt{3},0,0,\pm i/2,0)$
- $(x_+,x_-,y,z,w_+,w_-) = (0,0,\frac{\pm\sqrt{3-\ell^2}}{(6\sqrt{2})},\ell/6,0,0)$
- $(x_+,x_-,y,z,w_+,w_-) = (\frac{\ell^2-1}{\ell^2+8},-\sqrt{3}\frac{\ell^2-1}{\ell^2+8},\pm\frac{\sqrt{12-3\ell^2}}{2(\ell^2+8)},\frac{3\ell}{2(\ell^2+8)},0,\pm\frac{\sqrt{(\ell^2-1)(\ell^2-4)}}{2(\ell^2+8)})$
- $(x_+,x_-,y,z,w_+,w_-) = (\frac{\ell^2-1}{\ell^2+8},\sqrt{3}\frac{\ell^2-1}{\ell^2+8},\pm\frac{\sqrt{12-3\ell^2}}{2(\ell^2+8)},\frac{3\ell}{2(\ell^2+8)},\pm\frac{\sqrt{(\ell^2-1)(\ell^2-4)}}{2(\ell^2+8)},0)$

Chapitre 4

Isotropisation of Bianchi class A models with a minimally coupled scalar field and a perfect fluid

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Abstract

We look for the necessary conditions allowing the Universe isotropisation in presence of a minimally coupled and massive scalar field with a perfect fluid. We conclude that it arises only when the Universe is scalar field dominated, leading to flat spacelike sections and accelerated expansion, and examine the case of a SUGRA theory.

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4.1 Introduction

Today the introduction of scalar fields in cosmology obeys major reasons taking roots in the extension of the standard particle physics model such as supersymmetry which requires additional degrees of freedom represented by these fields. In a general way, most of the theories predicting extra dimensions at high energy could generate scalar fields[113] via compactification processes. Supergravity theory(SUGRA)[179, 180] related to supersymmetry concept or Higgs mechanism which allows us to explain the mass of particles also imply some scalar fields. From an observational point of view they could be responsible for dark matter[136, 181, 147] as well as dark energy[9, 10, 182, 183] although other explanations exist. They could also solve the so-called cosmological constant problem: most of these scalar fields are massive and thus able to mimic a variable cosmological constant.

Let us speak about the geometrical context of this paper. There exist nine anisotropic cosmological models classified by Bianchi in 1897. We will be interested in the curved Bianchi class A models since we have studied the spatially flat Bianchi type I model in [109] and there is no adapted ADM Hamiltonian formulation for the Bianchi class B models. Among the Bianchi class A models, the Bianchi type IX one contains the solutions of the positively curved isotropic FLRW model and the Bianchi type II one characterizes the strong anisotropic phases[72]. Unless we assume a Universe born isotropic and homogeneous, as instance thanks to a quantum principle selecting this type of particular model among all the possible ones, it is legitimate to ask why our Universe is so symmetric. It seems more natural to suppose that it was initially less symmetric and that it asymptotically evolves to an FLRW model. This is one of the reasons why the study of anisotropic models is so important. It allows us to explore the mechanisms responsible for the isotropisation of our Universe and to put some constraints that may be compared to observations on its final isotropic state. Moreover, from the initial state point of view, the oscillatory approach of the singularity by

the Bianchi type IX model is generally considered as more generic than the one of the FLRW models and could be shared by the most general inhomogeneous models as conjectured by Belinskij, Khalatnikov and Lifchitz [184, 185].
Our goal is to find some scalar field properties allowing the Universe to reach isotropy and then the dynamical behaviours of the metric and potential. In [127], we have shown that isotropisation of curved class A Bianchi models in presence of a massive scalar field but without a perfect fluid always leads to a late times acceleration which is not necessary the case when there is no curvature[105]. In [109], we have seen that in presence of a perfect fluid, the isotropisation of the flat Bianchi type I model leads to a decelerated expansion if asymptotically the difference $p_\phi - \rho_\phi$ between the pressure and the density of the scalar field is proportional to the density ρ of the perfect fluid. What happens when we consider both curvature and perfect fluid? Here, we will try to answer this question.
To this end, we will use the ADM Hamiltonian formalism[78] to get a first order equations system that we will study by help of dynamical systems analysis[25]. Most of times, dynamical analysis of the field equations in cosmology rest on the orthonormal frame formalism and Hubble-normalized variables as shown in Wainwright and Ellis book[25]. It allows us to study a large number of cosmological models in various situations, even the most complex one such as the inhomogeneous cosmologies[186] or the presence of magnetic fields[187], finding and classifying all the equilibrium points of these systems. Some scalar-tensor theories have also been studied in this way but, to our knowledge, their forms were always completely specified, i.e. they did not contain any unspecified function of the scalar field. Here, we want to consider a class of scalar-tensor theories containing two unspecified functions of the scalar field and just look for the stable isotropic state the Universe can reach. Hence, we aim to study a larger class of scalar-tensor theories than usually and it is one of the reasons why we have not used the powerful orthonormal frame formalism but rather the more traditional Hamiltonian ADM formalism which have proved to be useful in such a case[42].
The plan of this work is as follows: in the second part we establish the Hamiltonian field equations and, after having remembered the results we obtained without a perfect fluid, we study the isotropisation process when it is present. We discuss the physical meaning of our results in the last section.

4.2 Field equations and dynamical analysis

In the first subsection, we derive the Hamiltonian field equations and in the second one, we use dynamical systems analysis to study the stable isotropic states.

4.2.1 Field equations

We will use the following metric, reflecting the 3+1 decomposition of spacetime:

$$ds^2 = -(N^2 - N_i N^i)d\Omega^2 + 2N_i d\Omega\omega^i + R_0^2 g_{ij}\omega^i\omega^j \tag{4.1}$$

The ω_i are the 1-forms generating the Bianchi homogeneous spaces, N and N_i are the lapse and shift functions and g_{ij} are the metric functions parameterised by Misner[24] as:

$$\begin{aligned} g_{11} &= e^{-2\Omega+\beta_+ + \sqrt{3}\beta_-} \\ g_{22} &= e^{-2\Omega+\beta_+ - \sqrt{3}\beta_-} \\ g_{33} &= e^{-2\Omega-2\beta_+} \end{aligned}$$

The $\beta_\pm$ functions describe the anisotropy whereas Ω is the metric isotropic part. The action of the minimally coupled and massive scalar field theory with a non tilted perfect fluid writes:

$$S = (16\pi)^{-1}\int \left[R - (3/2+\omega(\phi))\phi^{,\mu}\phi_{,\mu}\phi^{-2} - U(\phi) + 16\pi c^4 L_m\right]\sqrt{-g}d^4x \tag{4.2}$$

U is the potential of the scalar field ϕ whose coupling with the metric is described by the Brans-Dicke coupling function ω[1]. L_m is the Lagrangian of the non tilted perfect fluid whose equation of state is $p = (\gamma-1)\rho$ with $\gamma \in [1,2]$. It describes a dust fluid when $\gamma = 1$ and a radiative fluid when $\gamma = 4/3$. The other important value is $\gamma = 0$ and corresponds to a cosmological constant which has been discussed in

1. A scalar field transformation sometimes allows to reduce the two unspecified functions ω and U to a single function. However, the transformation is not always analytically possible and it is why it is more general to consider the two functions.

[127]. Technical details allowing to get the ADM Hamiltonian from the action (4.2) have been given in [125, 77, 105]. Hence we write directly:

$$H^2 = p_+^2 + p_-^2 + 12\frac{p_\phi^2\phi^2}{3+2\omega} + 24\pi^2 R_0^6 e^{-6\Omega}U + \delta e^{3(\gamma-2)\Omega} + V(\Omega,\beta_+,\beta_-) \quad (4.3)$$

with $p_\pm$ and p_ϕ, respectively the conjugate momenta of the $\beta_\pm$ variables and the scalar field. $V(\Omega,\beta_+,\beta_-)$ is the curvature potential characterising each curved Bianchi class A model and given in table 4.1. δ is a

Type	Expression of $V(\Omega,\beta_+,\beta_-)$
II	$24\pi^2 R_0^4 e^{-4\Omega+4\beta_+ +\sqrt{3}\beta_-}$
VI_0	$24\pi^2 R_0^4 e^{-4\Omega+4\beta_+}(\cosh 4\sqrt{3}\beta_- + 1)$
VII_0	$24\pi^2 R_0^4 e^{-4\Omega+4\beta_+}(\cosh 4\sqrt{3}\beta_- - 1)$
$VIII$	$24\pi^2 R_0^4 e^{-4\Omega}[e^{4\beta_+}(\cosh 4\sqrt{3}\beta_- - 1) + 1/2e^{-8\beta_+} + 2e^{-2\beta_+}\cosh -2\sqrt{3}\beta_-]$
IX	$24\pi^2 R_0^4 e^{-4\Omega}[e^{4\beta_+}(\cosh 4\sqrt{3}\beta_- - 1) + 1/2e^{-8\beta_+} - 2e^{-2\beta_+}\cosh -2\sqrt{3}\beta_-]$

TAB. 4.1 – *Curvature potentials for Bianchi type II, VI_0, VII_0, $VIII$ and IX models*

positive constant proportional to $(\gamma-1)\rho_0$. Using (4.3), the Hamiltonian equations are:

$$\dot\beta_\pm = \frac{\partial H}{\partial p_\pm} = \frac{p_\pm}{H} \quad (4.4)$$

$$\dot\phi = \frac{\partial H}{\partial p_\phi} = \frac{12\phi^2 p_\phi}{(3+2\omega)H} \quad (4.5)$$

$$\dot p_\pm = -\frac{\partial H}{\partial \beta_\pm} = -\frac{\partial V}{2H\partial\beta_\pm} \quad (4.6)$$

$$\dot p_\phi = -\frac{\partial H}{\partial \phi} = -12\frac{\phi p_\phi^2}{(3+2\omega)H} + 12\frac{\omega_\phi\phi^2 p_\phi^2}{(3+2\omega)^2 H} - 12\pi^2 R_0^6\frac{e^{-6\Omega}U_\phi}{H} \quad (4.7)$$

$$\dot H = \frac{dH}{d\Omega} = \frac{\partial H}{\partial\Omega} = -72\pi^2 R_0^6\frac{e^{-6\Omega}U}{H} + 3/2\delta(\gamma-2)\frac{e^{3(\gamma-2)\Omega}}{H} + \frac{\partial V}{2H\partial\Omega} \quad (4.8)$$

In this paper, we will choose $N_i = 0$, i.e. a diagonal metric, and we derive[78] that

$$N = \frac{12\pi R_0^3 e^{-3\Omega}}{H}$$

Now, we have to rewrite these equations with some variables, bounded in the neighbourhood of the isotropy. In [127] we had used the following variables common to all the curved Bianchi class A models:

$$x_\pm = p_\pm H^{-1} \quad (4.9)$$

$$y = \pi R_0^3\sqrt{U}e^{-3\Omega}H^{-1} \quad (4.10)$$

$$z = p_\phi\phi(3+2\omega)^{-1/2}H^{-1} \quad (4.11)$$

and ϕ the scalar field. These variables are real as long as $U > 0$ and $3+2\omega > 0$ which is necessary to respect the weak equivalence principle. Each of them has a physical interpretation:

– $x_\pm^2$ are proportional to the shear parameters $\Sigma_\pm$ defined in [25].
– y^2 is proportional to $(\rho_\phi - p_\phi)/(d\Omega/dt)^2$, $(d\Omega/dt)^2$ being the Hubble variable when the Universe is isotropic, ρ_ϕ and p_ϕ the density and pressure of the scalar field.
– z^2 is proportional to $(\rho_\phi +_\phi)/(d\Omega/dt)^2$, $(d\Omega/dt)^2$.
– We deduce from these two last points that the density parameter Ω_ϕ for the scalar field is a linear combination of y^2 and z^2 or, when the scalar field is quintessent, that these two variables are proportional to Ω_ϕ.

We had also defined some "w" variables characterising the curvature of each Bianchi model and which are shown in the table 4.2. They are related to the three N_i variables describing the curvature in the paper of Horwood and Wainwright[126] or in the book edited by Wainwright and Ellis[25] and defined by using a symmetry group structure. In this last book, the curvature of the Bianchi type II model, VI_0 and VII_0 models, $VIII$ and IX models are respectively described by N_1 , (N_2,N_3) and (N_1,N_2,N_3) variables. Here,

Bianchi models	Associated variables
II	$w = \pi R_0^2 e^{-2\Omega+2(\beta_+ +\sqrt{3}\beta_-)} H^{-1}$
VI_0 and VII_0	$w_\pm = \pi R_0^2 e^{-2\Omega+2(\beta_+ \pm\sqrt{3}\beta_-)} H^{-1}$
$VIII$ and IX	$w_p = \pi R_0^2 e^{-2\Omega+2\beta_+} H^{-1}$ $w_m = \pi R_0^2 e^{-2\Omega-2\beta_+} H^{-1}$ $w_- = e^{2\sqrt{3}\beta_-}$

TAB. 4.2 – w *variables characterising he curvature of each Bianchi model.*

in a similar way, we could redefine three variables w_i, $i = 1,2,3$ such that for the Bianchi type II model, VI_0 and VII_0 models, $VIII$ and IX models, the curvature be described by $(w_1 = w)$, $(w_1 = w_+, w_2 = w_-)$ and $(w_1 = w_p w_-, w_2 = w_p/w_-, w_3 = w_m)$, thus recovering the same unified picture as in [25].
In this paper, we will also consider an additional variable called k and related to the presence of a perfect fluid. It is defined by

$$\begin{aligned} k^2 &= \delta e^{3(\gamma-2)\Omega} H^{-2} \\ k^2 &= \delta y^2 V^{-\gamma} U^{-1} \end{aligned} \tag{4.12}$$

k is proportional to the density parameter of the perfect fluid, one of the main parameters in cosmology. It can be shown by checking that $k^2 \propto V^{-\gamma}/(\frac{d\Omega}{dt})^2$. k is not independent from the other variables and when no perfect fluid is considered, $k = 0$ strictly.
For each Bianchi model, we have rewritten the Hamiltonian constraint and the field equations with these variables in the second appendix.

4.2.2 Isotropisation

In the first subsection, we define the different ways to reach a stable isotropic state. In the second one, we recall our results obtained without the perfect fluid. In the third one, we discuss about their stability. In the fourth one, we extend them by considering the presence of a perfect fluid.

Different kinds of isotropisation

In [127] when no perfect fluid is present, we had defined the isotropy as the convergence of the metric functions to a common form such as the Hubble parameter is the same in any directions. It implied $d\beta_\pm/dt \to 0$ and $\beta_\pm \to const$ and thus that it should arise when $p_\pm e^{3\Omega} \to 0$. This definition is unchanged in presence of a perfect fluid.
Different kind of isotropisation may exist, leading to a forever expanding model, a singularity or a static Universe. We had shown that when there is no perfect fluid, isotropy only occurs when $\Omega \to -\infty$ and $x \to 0$, i.e. for a forever expanding Universe. Looking at the field equations, we find three ways to reach an isotropic stable state that we have classified in three classes:

1. Class 1: all the variables but not necessarily the scalar field reach equilibrium with $y \neq 0$.
2. Class 2: all the variables but not necessarily the scalar field reach equilibrium with $y = 0$.
3. Class 3: all the variables do not reach equilibrium but $x_\pm$ which, as the w functions, have to vanish.

For the class 2, generally nothing can be deduced about the asymptotic behaviours of the metric functions and potential. It has been numerically observed in a paper in preparation where a non minimally coupling between a scalar field and a perfect fluid is considered. For the class 3, y and z do not necessarily need to reach equilibrium when the Universe isotropises. They just have to be bounded when $\Omega \to -\infty$, implying that they oscillate. Hence, the signs of their derivatives, which do not asymptotically vanish [2], change continuously [3]. We have numerically observed the class 3 isotropisation in presence of several scalar fields[116] and it seems to be associated to an oscillating behaviour of ℓ.
In this paper, we will study the class 1 isotropisation. In the two next subsections, we briefly recall the results we obtained in [127] without a perfect fluid and then discuss the assumptions we have made related to their stability.

2. We are assuming that they do not reach equilibrium!
3. The variables are bounded

Without the perfect fluid

The results we obtained in [127] are the following. We define the function ℓ of the scalar field:

$$\ell = \phi U_\phi U^{-1}(3+2\omega)^{-1/2}$$

The equilibrium points corresponding to class 1 isotropisation are given by $(x_\pm,y,z) = (0,\pm\sqrt{3-\ell^2}(6\sqrt{2})^{-1},\ell/6)$, the w variables related to the curvature (see table 4.1) being zero. Our conclusion about isotropisation, valid whatever the curved Bianchi class A models when no perfect fluid is present, was that it occurs for a forever expanding Universe ($\Omega \to -\infty$) and it requires ℓ^2 to tend to a constant ℓ_0 smaller than 1. Then the metric functions tend to $t^{\ell_0^{-2}}$ if $\ell_0 \neq 0$ or to a De Sitter model otherwise. The Universe is thus asymptotically accelerated and flat. The scalar field asymptotical behaviour may be determined by the asymptotical solution of the first degree differential equation $\dot{\phi} = 2\phi^2 U_\phi (3+2\omega)^{-1}U^{-1}$.
For Bianchi type II, VI_0 and VII_0 model, isotropisation will occur at late times if the Hamiltonian H is initially positive and at early times otherwise. It is easily shown by noting that H is a monotonic function of Ω with a constant sign. Then, using the relation $dt = -Nd\Omega$, it comes that Ω is a decreasing(increasing) function of the proper time t when H is positive (negative). Since the Universe only isotropises in $\Omega \to -\infty$, it thus corresponds to late times and forever expanding Universe. For the Bianchi types $VIII$ and IX models, it is not possible to show that H is a monotonic function and thus, the isotropisation time is undetermined.

Stability of our results

The above results or the ones of the present paper are the determination of the isotropic equilibrium points, some necessary conditions for isotropisation and the asymptotical behaviours of some functions in the neighbourhood of these points. However the asymptotical behaviours are determined by calculating the exact solutions for each equilibrium point and they will be correct only if on one hand ℓ and in the other hand the variables (y,z,w) (and k when we will consider a perfect fluid) tend sufficiently fast to their equilibrium values. Otherwise, they will be different. Let us explain why.

The first kind of instability comes from ℓ. As instance, when we look for $x_\pm$ asymptotical behaviours, we need to calculate $exp(\int \ell^2 d\Omega)$, with ℓ^2 asymptotically tending to a constant ℓ_0 (vanishing or not) near the isotropic state. In our calculation, we have assumed that asymptotically when $\Omega \to -\infty$, $exp(\int \ell^2 d\Omega) \to exp(\ell_0^2\Omega)$ but this is true only if ℓ^2 tends sufficiently fast to its constant equilibrium value. As instance, if $\ell^2 \to \ell_0 + \Omega^{-1/2}$, it is different from a pure exponential. Hence, our results will be valid as long as the following assumption holds:

– When ℓ tends to a constant ℓ_0 (vanishing or not) such that $\ell^2 \to \ell_0^2 + \delta\ell^2$, $\int(\ell_0^2+\delta\ell^2)d\Omega \to \ell_0^2\Omega + const.$

If it is not true, the asymptotical behaviours for the metric functions (and potential) are different from classical power or exponential laws. This problem could be overcame since our results allow to calculate $\phi(\Omega)$ and thus $\ell(\Omega)$. Hence, it should be easy to generalise them by keeping the $\int \ell^2 d\Omega$ term instead of considering that it tends to $\ell^2\Omega$ but then they would not be on a closed form.

The second kind of instability can not be solved so easily. In the same way, the asymptotical behaviours we have determined will be true only if the variables (y,z,w,k) tend sufficiently fast to their equilibrium values. As instance near isotropy we have $y \to \pm\sqrt{3-\ell^2}(6\sqrt{2})^{-1}$ and when we integrate the differential equation for $x_\pm$, we assume that $exp(\int y^2 d\Omega) \to exp(\int(3-\ell^2)/72 d\Omega)$. But once again, this is not exact if y^2 tends to its equilibrium value slower than Ω^{-1} and we have to make the same kind of assumption for (y,z,w,k) as for ℓ. For partly solve this problem, it would be necessary to consider some small perturbations of the exact solutions but until now we have not succeed to get any interesting results, even for the flat model.

To summarize, the results of this paper related to asymptotical behaviours will be valid for a class 1 isotropisation if the function ℓ and the variables (y,z,w,k) tend sufficiently fast to their equilibrium values or, more physically, if the Universe tends sufficiently fast to its isotropic state. The restriction on ℓ may be easily solved but the ones on (y,z,w,k) require a more careful examination. In the following subsection, we consider the isotropisation of a curved Bianchi class A model in presence of a perfect fluid, first when k vanishes and then when it tends to a non vanishing constant.

With a perfect fluid

Depending on the vanishing of k near an isotropic equilibrium state, the results summarize in the section 4.2.2 will or will not be modified.

$k \to 0$

When $k \to 0$ near isotropy, the results are the same as those found when we consider no perfect fluid. In particular, isotropisation always arise for a forever expanding Universe, i.e. when $\Omega \to -\infty$. Obviously, we find the same equilibrium points and assuming that k tends sufficiently fast to its equilibrium value(see section 4.2.2), we also recover the same asymptotical behaviours. However, the limit $k \to 0$ plays the role of a new constraint. This fact was noted in [116] for the flat Bianchi type I model. In this last paper we had shown that the interval of ℓ allowing for isotropy was smaller when we consider a perfect fluid such that $k \to 0$ than without it: in this last case isotropy requires $\ell^2 < 3$, otherwise $\ell^2 < 3/2\gamma$[4].
Does the limit $k \to 0$ also change the necessary conditions for isotropy when we consider some curvature? Near equilibrium, the w variables have to vanish and are proportional to $e^{-2\Omega}H^{-1}$. But $e^{-2\Omega}$ diverges and thus, since $w \to 0$, H have to be larger than $e^{-2\Omega}$, i.e.:

$$H >> e^{-2\Omega}$$

Moreover, we have $y^2 = Ue^{-2\Omega}e^{-4\Omega}H^{-2}$ and since near isotropy y tends to a non vanishing constant whereas $e^{-4\Omega}H^{-2}$ tends to vanish, we deduce that

$$U >> e^{2\Omega} >> V^{-\gamma}$$

and then from (4.12) that $k \to 0$. Consequently, starting from the fact that $w \to 0$, we conclude that $k \to 0$ without any modification of the necessary condition for isotropisation on the contrary from the Bianchi type I model. Moreover, it means that the energy density ρ_ϕ of the scalar field and its pressure p_ϕ are such that $U \propto p_\phi - \rho_\phi >> \rho_m$: the Universe is dynamically dominated by the scalar field. Thus, the results obtained in the vacuum (i.e. $k = 0$ strictly) are not changed when we consider a perfect fluid such as $k \to 0$.

$k \not\to 0$

Now, we consider what happens when $k \not\to 0$. The necessary condition for isotropy is still $p_\pm e^{3\Omega} \to 0$ and we have to determine if it occurs for a forever expanding, contracting or static Universe.

- If it arises for a diverging Ω, it means that at equilibrium, we must have $x_\pm \to 0$ as explained in the section 4.2.2.
- If it arises for a finite value of Ω, then we must have $p_\pm \to 0$.
  - Let us assume that in the same time $x_\pm \not\to 0$. Since $p_\pm \to 0$, from (4.9) we deduce that H have to vanish otherwise $x_\pm \to 0$. But then k^2, which is proportional to the perfect fluid density parameter, diverges and the constraint is not respected because, near isotropy, all the variables have to be bounded as shown in [127]. Hence, H can not tend to zero and $x_\pm$ must vanish near equilibrium.
  - In the same way, when Ω tends to a non vanishing constant, H can not diverge because then $k \to 0$ which is not in agreement with the assumption of this subsection.

  Consequently, when the isotropy occurs for a finite Ω, the Hamiltonian have to tend to a bounded and non vanishing quantity and it is thus the same for the w variables.

To summarize, in the neighbourhood of the isotropic state:

- If Ω diverges, the equilibrium points are such that $x_\pm \to 0$.
- If $\Omega \to const \neq 0$, the equilibrium points are such that $x_\pm \to 0$ and w variables are non vanishing and bounded.

Whatever the curved Bianchi models, the only equilibrium points corresponding to these requirements when solving the field equations[5] are defined by $(x_\pm, y, z) = (0, \pm\frac{\sqrt{\gamma(2-\gamma)}}{4\sqrt{2\pi R_0^3}\ell}, \frac{\gamma}{4\ell})$, the w variables related to the curvature(see table 4.2) being 0[6]. The Hamiltonian constraint implies that $k^2 = 1 - \frac{3\gamma}{2\ell^2}$ and consequently $\ell^2 > \frac{3}{2}\gamma$. This inequality is independent from any assumption on how far the isotropic state is reached. As

4. This inequality rests on the asymptotical behaviour of k and, as discussed in section 4.2.2, it may vary if k does no tend suffi ciently fast to its equilibrium value. However, the limit $\ell^2 < 3$ have always to be respected since it is required for the existence of the equilibrium points, independently on how fast the isotropic state is reached.

5. For the Bianchi type $VIII$ and IX models where the equations are far from being simple, it is not possible to solve them directly. We proceed by putting $x_\pm = 0$ and $w_- = 1$ in the equations for $x_\pm$, these values being these required for isotropy. We then show that $x_\pm$ can reach equilibrium only if w_p and w_m vanish which allow us to determine the equilibrium values for the other variables. Since w_p and w_m tend to vanish, the Hamiltonian constraint shows that all the variables are bounded. For the other Bianchi models, we can show in the same way as in [127] that all the variables are bounded near the isotropic equilibrium state.

6. There exist some other equilibrium points for which k or ℓ may be chosen such that $x_\pm = 0$ and the constraint be respected but they correspond to complex values of some variables.

the w variables are vanishing, it follows that Ω must diverge and not tend to a constant. Consequently, we calculate that asymptotically the Hamiltonian, whose form is given in the second appendix as a function of the variables, behaves as:

$$H \to e^{-\frac{3}{2}(2-\gamma)\Omega}$$

This is in agreement with the limit $k^2 \to const \neq 0$ and the definition for k which also implies that $U \propto V^{-\gamma}$. Hence, the scalar field plays the same dynamical role as the perfect fluid and we can show that their energy densities scales in the same way, preventing any accelerated expansion. We also get that the w variables(but w_- which tends to a non vanishing constant for the Bianchi type $VIII$ and IX variables) all behave as:

$$w \to e^{(1-\frac{3\gamma}{2})\Omega}$$

For the considered range of γ, we derive that $w \to 0$ only if $\Omega \to +\infty$. But for the $x_\pm$ variables, it comes:

$$x_\pm \to x_0 e^{(2-3\gamma)\Omega}(e^{(1+\frac{3\gamma}{2})\Omega} + x_1)$$

x_0 being an integration constant. It follows that if $\gamma \in$ [1,2] and $\Omega \to +\infty$, x diverges. Consequently, the isotropic state can not be reached for this range of γ. Knowing $x_\pm$ and H, we calculate that:

$$p_\pm \to e^{-\frac{1}{2}(2+3\gamma)\Omega} + cte$$

Hence, $p_\pm e^{3\Omega}$, the w and $x_\pm$ variables vanish only if $\gamma < 2/3$ and $\Omega \to -\infty$. Then, we find that $e^{-\Omega} \to t^{\frac{2}{3\gamma}}$ and, from the definition of y and the property $U \propto V^{-\gamma}$, we derive that $U \to t^{-2}$. This restriction on γ does not exist for the flat Bianchi type I model[109, 116] and does not fit an ordinary perfect fluid such that $\gamma \in$ [1,2].

4.3 Conclusion

In this work, we have determined the necessary but not sufficient conditions for class 1 isotropisation of curved Bianchi class A models when a minimally and massive scalar field with a perfect fluid are considered. We have assumed that $U > 0$ and $3 + 2\omega > 0$ such that the weak energy principle is respected. Moreover, some of our results related to asymptotical behaviours are valid as long as the isotropic state is reached sufficiently fast.

We can distinguish two cases depending on the vanishing of k, a variable proportional to the perfect fluid density parameter Ω_m. When isotropy occurs with $k \to 0$, we have thus $\Omega_m \to 0$, $U \propto p_\phi - \rho_\phi > \rho_m$ and the results are the same as in [127] where no perfect fluid is present:

Class 1 isotropisation with $\Omega_m \to 0$:
A necessary condition for isotropisation of curved Bianchi class A models in presence of a minimally and massive scalar field such that $\Omega_m \to 0$ will be that the quantity $\ell = \phi U_\phi U^{-1}(3+2\omega)^{-1/2}$ tends to a constant ℓ_0, whose square is smaller than one. For the Bianchi type II, VI_0 and VII_0 models, it arises at late (early) times if the Hamiltonian is initially positive(negative). For the Bianchi type $VIII$ and IX models, the time of isotropisation is undetermined. If $\ell_0 \neq 0$, the metric functions tend to a power law $t^{\ell_0^{-2}}$ and the potential vanishes as t^{-2}. If $\ell_0 = 0$, the Universe tend to a De Sitter model and the potential to a constant. The isotropisation process always leads to a flat and accelerated Universe.

Considering the limit near isotropy of the Hamiltonian equation for $\dot{\phi}$ rewritten with the normalised variables (see appendice), we deduce that the scalar field asymptotically behaves as the limit of the solution for

$$\dot{\phi} = 2\phi^2 U_\phi U^{-1}(3+2\omega)^{-1}$$

as $\Omega \to -\infty$, in the same way as in [127]. This last equation allows us to deduce the asymptotical behaviour of $\ell(\Omega)$ when we specify ω and U. The second result of this work concerns the case for which k, or equivalently Ω_m, tends to a non vanishing constant implying that $U \propto p_\phi - \rho_\phi \propto \rho_m$. We have then:

Class 1 isotropisation with $\Omega_m \to const \neq 0$:
The isotropisation of curved Bianchi class A models in presence of a minimally and massive scalar field such that $\Omega_m \to const \neq 0$ is impossible if the perfect fluid is an ordinary one such that $\gamma \in$ [1,2]. It will only occur if $\gamma < 2/3$, which generally corresponds to a quintessent fluid equation of state.

Now, we examine these results with respect to supergravity. In [179, 180], it is shown that quintessence theories should be based on supergravity. A scalar tensor theory is then derived, defined by $\omega + 3/2 = \phi^2$ and $U = \Lambda^{4+m}\phi^{-m}e^{\frac{n}{2}\phi^2}$. It is able to solve the coincidence problem and even the fine tuning problem if $m \geq 11$. What about class 1 isotropisation? We calculate that:

$$\ell^2 = (\frac{n\phi^2 - m}{\sqrt{2}\phi})^2$$

and when no matter is present or if $k \to 0$, the scalar field asymptotically behaves as:

$$\phi \to \pm\sqrt{\frac{m - \phi_0 e^{2n\Omega}}{n}}$$

ϕ_0 being an integration constant.
When $n > 0$, $\phi \to (m/n)^{1/2}$ implying that m should be positive. $\ell \to 0$ and the necessary conditions for isotropisation are thus respected. If it arises, the Universe tends to a De-Sitter model. It could thus describe the inflationary period, before the domination of the matter. This case is plotted on figure 4.1 for the Bianchi type IX model.
When $n < 0$, the scalar field behaves as $\phi \to \pm\sqrt{\frac{-\phi_0 e^{2n\Omega}}{n}}$. It is defined when $\Omega \to -\infty$ if $\phi_0 > 0$ and

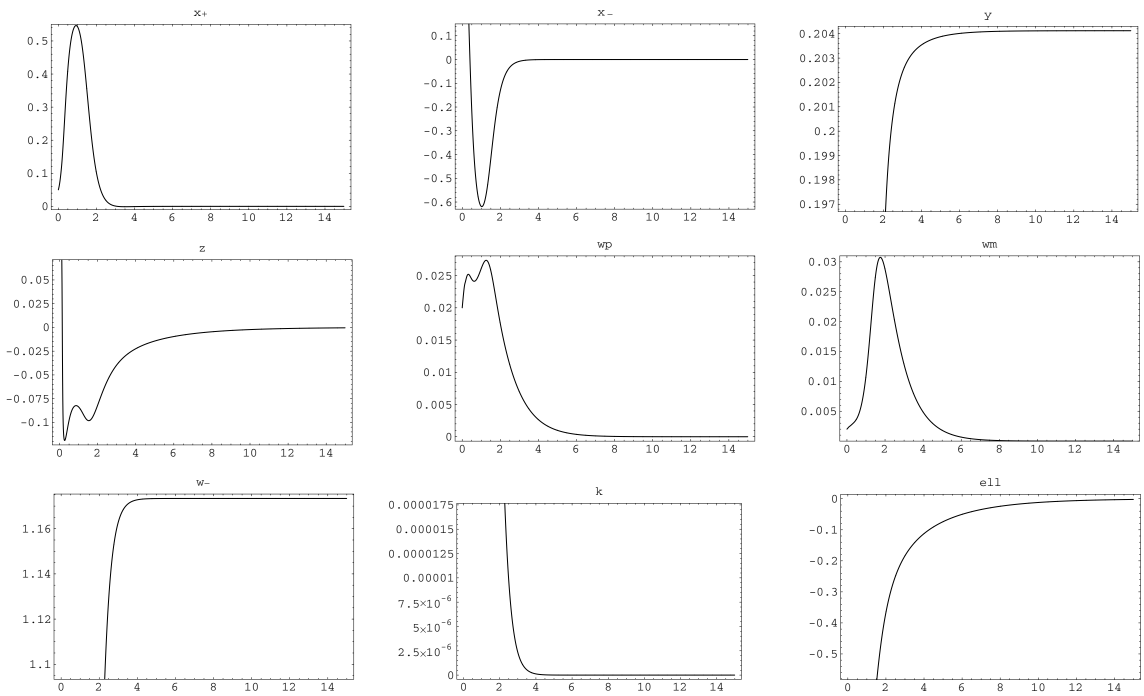

FIG. 4.1 – *Isotropisation of SUGRA theory for Bianchi type IX model. Initial conditions and potential parameters respectively are* $(x_+,x_-,y,z,w_p,w_m,w_-,\phi) = (0.05,0.83,0.025,0.12,0.02,0.002,0.2,0.14)$ *and* $(\Lambda,m,n) = (2.0,1.1,0.15)$.

then diverges. It follows that ℓ also diverges and thus a class 1 isotropisation is not possible as confirmed by numerical simulations.
Summarising, if the Universe isotropises, this theory issued from SUGRA leads an anisotropic curved Universe to a flat isotropic De Sitter one dominated by the scalar field. It could be a good description for an inflationary period. Numerical simulations have not shown any class 2 or 3 isotropisation.

In conclusion, we knew that when no perfect fluid is present, the class 1 isotropisation of an anisotropic curved Universe may lead the Universe to flat spacelike sections and accelerated expansion if some necessary conditions are respected. The question was: does this acceleration, due to the presence of curvature, always exist in presence of a perfect fluid. The answer is "yes" when the density parameter of the perfect fluid asymptotically vanishes. Then, its presence does not change the asymptotic isotropic state or the necessary conditions to reach it. Contrary to the flat Bianchi type I model for which an isotropic state such that $\Omega_m \to const \neq 0$ may exist, the perfect fluid and the scalar field playing the same dynamical role, the isotropic state in presence of curvature is always scalar field dominated but if the perfect fluid is an exotic one. Future research should be concerned by a scalar field which violates the energy conditions, i.e. such that $\omega < -3/2$ or $U < 0$ or which is not minimally coupled to a perfect fluid. This last possibility, which would allow to extend our results to the Hyperextended Scalar Tensor theory (i.e. with a varying gravitation function) with a perfect fluid, is currently under consideration in a paper in preparation.

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4.4 Appendix: field equations of the curved Bianchi models with normalised variables

Bianchi type II
The Hamiltonian constraint writes:

$$x_+^2 + x_-^2 + 24y^2 + 12z^2 + 12w^2 + k^2 = 1 \tag{4.13}$$

The Hamiltonian equations become:

$$\dot{x}_+ = 72y^2 x_+ + 24w^2 x_+ - 24w^2 - 3/2(\gamma - 2)k^2 x_+ \tag{4.14}$$

$$\dot{x}_- = 72y^2 x_- + 24w^2 x_- - 24\sqrt{3}w^2 - 3/2(\gamma - 2)k^2 x_- \tag{4.15}$$

$$\dot{y} = y(6\ell z + 72y^2 - 3 + 24w^2) - 3/2(\gamma - 2)k^2 y \tag{4.16}$$

$$\dot{z} = y^2(72z - 12\ell) + 24w^2 z - 3/2(\gamma - 2)k^2 z \tag{4.17}$$

$$\dot{w} = 2w(x_+ + \sqrt{3}x_- + 12w^2 + 36y^2 - 1) - 3/2(\gamma - 2)k^2 w \tag{4.18}$$

To get an autonomous system, we need a first order equation for ϕ. Rewriting (4.5), it comes:

$$\dot{\phi} = 12\frac{z\phi}{\sqrt{3 + 2\omega}} \tag{4.19}$$

This equation is the same for any Bianchi models. The equation for $\dot{H}$ may be rewritten as:

$$\dot{H} = -H(72y^2 + 24w^2 + \frac{3}{2}(\gamma - 2)k^2) \tag{4.20}$$

Bianchi VI_0 and VII_0 models
The Hamiltonian constraint writes:

$$x_+^2 + x_-^2 + 24y^2 + 12z^2 + 12(w_+ \pm w_-)^2 + k^2 = 1 \tag{4.21}$$

and the Hamiltonian equations become:

$$\dot{x}_+ = 72y^2 x_+ + 24(x_+ - 1)(w_- \pm w_+)^2 - 3/2(\gamma - 2)k^2 x_+ \tag{4.22}$$

$$\dot{x}_- = 72y^2 x_- + 24x_-(w_- \pm w_+)^2 + 24\sqrt{3}(w_-^2 - w_+^2) - 3/2(\gamma - 2)k^2 x_- \tag{4.23}$$

$$\dot{y} = y(6\ell z + 72y^2 - 3 + 24(w_- \pm w_+)^2) - 3/2(\gamma - 2)k^2 y \tag{4.24}$$

$$\dot{z} = y^2(72z - 12\ell) + 24z(w_- \pm w_+)^2 - 3/2(\gamma - 2)k^2 z \tag{4.25}$$

$$\dot{w}_+ = 2w_+\left[x_+ + \sqrt{3}x_- + 12(w_- \pm w_+)^2 + 36y^2 - 1\right] - 3/2(\gamma - 2)k^2 w_+ \tag{4.26}$$

$$\dot{w}_- = 2w_-\left[x_+ - \sqrt{3}x_- + 12(w_- \pm w_+)^2 + 36y^2 - 1\right] - 3/2(\gamma - 2)k^2 w_- \tag{4.27}$$

The equation for $\dot{H}$ is:

$$\dot{H} = -H\left[72y^2 + 24(w_+ \pm w_-)^2 + \frac{3}{2}(\gamma - 2)k^2\right] \tag{4.28}$$

<u>Bianchi $VIII$ and IX models</u>
The Hamiltonian constraint writes:

$$x_+^2 + x_-^2 + 24y^2 + 12z^2 + 12[w_p^3(1+w_-^4) \pm 2w_-(w_m w_p)^{3/2}(1+w_-^2) + w_-^2(w_m^3 - 2w_p^3)](w_-^2 w_p)^{-1} + k^2 = 1$$

and the Hamiltonian equations are:

$$\dot{x}_+ = 72y^2 x_+ + 24\{w_p^3(x_+ - 1)(1+w_-^4) \pm w_-(1+2x_+)(w_m w_p)^{3/2}(1+w_-^2) + w_-^2\left[(2+x_+)w_m^3 - 2(x_+ - 1)w_p^3\right]\}(w_-^2 w_p)^{-1} - 3/2(\gamma-2)k^2 x_+ \quad (4.29)$$

$$\dot{x}_- = 72y^2 x_- + 24\{w_p^3\left[w_-^4(x_- - \sqrt{3}) + x_- + \sqrt{3})\right] \pm w_-(w_m w_p)^{3/2}[w_-^2(-\sqrt{3} + 2x_-) + (\sqrt{3} + 2x_-)] + w_-^2 x_-(w_m^3 - 2w_p^3)\}(w_-^2 w_p)^{-1} - 3/2(\gamma-2)k^2 x_- \quad (4.30)$$

$$\dot{y} = y\{6\ell z + 72y^2 - 3 + 24[w_p^3(1+w_-^4) \pm 2(w_m w_p)^{3/2} w_-(1+w_-^2) + w_-^2(w_m^3 - 2w_p^3)](w_-^2 w_p)^{-1}\} - 3/2(\gamma-2)k^2 y \quad (4.31)$$

$$\dot{z} = y^2(72z - 12\ell) + 24z[w_p^3(1+w_-^4) \pm 2(w_m w_p)^{3/2} w_-(1+w_-^2) + w_-^2(w_m^3 - 2w_p^3)](w_-^2 w_p)^{-1} - 3/2(\gamma-2)k^2 z \quad (4.32)$$

$$\dot{w}_p = w_p\{-2 + 2x_+ + 72y^2 + 24[w_p^3(1+w_-^4) \pm 2w_-(w_m w_p)^{3/2}(1+w_-^2) + w_-^2(w_m^3 - 2w_p^3)](w_-^2 w_p)^{-1}\} - 3/2(\gamma-2)k^2 w_p \quad (4.33)$$

$$\dot{w}_m = w_m\{-2 - 2x_+ + 72y^2 + 24[w_p^3(1+w_-^4) \pm 2w_-(w_m w_p)^{3/2}(1+w_-^2) + w_-^2(w_m^3 - 2w_p^3)](w_-^2 w_p)^{-1}\} - 3/2(\gamma-2)k^2 w_m \quad (4.34)$$

$$\dot{w}_- = 2\sqrt{3} w_- x_- \quad (4.35)$$

and

$$\dot{H} = -H[72y^2 + 24(\pm 2\frac{w_p^{1/2} w_m^{3/2}}{w_-} \pm 2w_p^{1/2} w_m^{3/2} w_- - 2w_p^2 + \frac{w_p^2}{w_-^2} + w_p^2 w_-^2 + \frac{w_m^3}{w_p}) + \frac{3}{2}(\gamma-2)k^2] \quad (4.36)$$

Chapitre 5

Isotropisation of flat homogeneous universes with scalar fields

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Abstract

Starting from an anisotropic flat cosmological model(Bianchi type I), we show that conditions leading to isotropisation fall into 3 classes, respectively 1, 2 ,3. We look for necessary conditions such that a Bianchi type I model reaches a stable isotropic state due to the presence of several massive scalar fields minimally coupled to the metric with a perfect fluid for class 1 isotropisation. The conditions are written in terms of some functions ℓ of the scalar fields. Two types of theories are studied. The first one deals with scalar tensor theories resulting from extra-dimensions compactification, where the Brans-Dicke coupling functions only depend on their associated scalar fields. The second one is related to the presence of complex scalar fields. We give the metric and potential asymptotical behaviours originating from class 1 isotropisation. The results depend on the domination of the scalar field potential compared to the perfect fluid energy density. We give explicit examples showing that some hybrid inflation theories do not lead to isotropy contrary to some high-order theories, whereas the most common forms of complex scalar fields undergo a class 3 isotropisation, characterised by strong oscillations of the ℓ functions.

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5.1 Introduction

If General Relativity is the best available theory describing our local Universe, it has recently become clear [9] that the modest amount of matter in the Universe (30% of the total energy density) is complemented by a large amount of exotic energy (70%). This exotic energy implies that the Universe is approximately spatially flat, and that its expansion is accelerating. To account for such a dynamics, several proposals exist which extend General Relativity to include higher order theories[170, 171], dissipative fluids[163, 172] or massive scalar fields. We are going to consider this last type of energy content.

Although most of papers only take into account one single scalar field, there are many reasons to consider the presence of several ones. Indeed, particle physics predicts high-order theories of gravity with extra-dimensions, which can be cast into an Einstein form in a 4-spacetime with several scalar fields by help of conformal transformations[112, 113, 26]. In supersymmetry, the adjunction of several scalar fields achieves equality between bosonic and fermionic degrees. Other reasons may be related to various inflationary mechanisms such as hybrid inflation, which needs two scalar fields[114, 115]: a first one, ψ, decreases to a local minimum corresponding to a false vacuum. Then the vacuum energy dominates and early time inflation begins. During this time, a second scalar field ϕ varies and when it reaches a threshold value ϕ_c, a fast variation of ψ arises. The two fields fit toward some values corresponding to a true vacuum and the end of inflation. A last reason could be the presence of complex scalar fields. A scalar tensor theory with one complex scalar field ζ can be cast into another one with two real scalar fields, ψ and ϕ, by help of the transformation $\zeta = \frac{1}{\sqrt{2}m}\psi e^{im\phi}$.

From a geometrical point of view, the standard cosmological model lies on the assumption that the Universe is perfectly isotropic, homogeneous and thus described by the FLRW metrics. However, they are very particular ones among the set of all possible metrics and we have to understand why our Universe may be described by them. One answer is to assume that it was not so symmetric at the beginning of time and that it quickly evolved to an isotropic and homogeneous state, as indicated by CMB observations. Moreover the singularity approach in FLRW models is far from being generic. Hence, it seems more natural to consider that the Universe was born with a more general geometry and has evolved toward a FLRW one. One possibility is to leave the isotropy hypothesis, keeping only homogeneity. Anisotropic models are described by the nine Bianchi models and allow studying how the Universe may tend to isotropy. Their behaviour near the singularity could be shared by inhomogeneous models[185, 184] and one of them admits the flat FLRW model solution consistent with recent CMB observations[188]: the flat Bianchi type I model.

The goal of this paper is to look for necessary conditions allowing for Bianchi type I model isotropisation when two minimally coupled and massive scalar fields with a perfect fluid are considered, and to study the asymptotic dynamics of the metric and potential in the neighbourhood of this state. From a technical point of view, we will use Hamiltonian ADM formalism giving the field equations as a first order differential system. We will rewrite it with normalised variables and look for equilibrium stable states corresponding to isotropic ones for the Universe[25]. Similar techniques have been used in presence of one single minimally coupled and massive scalar field[105] and with a perfect fluid[109]. For both cases, isotropic equilibrium points have been found corresponding to power or exponential law expansion for the metric functions. Here, we will also examine the stability of these results and Wald's cosmological "No Hair" theorem with respect to the presence of additional scalar fields. Intuitively, one could think that nothing should change and that the generalisation implying several scalar fields should be straightforward. However, we will see that it is not always the case and depends on the form of the scalar-tensor theory with respect to these fields.

The paper is organized as follows. In section 5.2 we derive the Hamiltonian equations and rewrite them with normalized variables. In section 5.3, we explain the assumptions we will use to study these equations. In section 5.4 we determine the equilibrium points, the monotonic functions and the asymptotic behaviour of the metric near equilibrium. We summarize and discuss our results in section 5.5. In section 5.6, some explicit applications are performed and we conclude in section 5.7.

5.2 Field equations

In this section we calculate the field equations. The metric of the Bianchi type I model is:

$$ds^2 = -(N^2 - N_iN^i)d\Omega^2 + 2N_i d\Omega\omega^i + R_0^2 g_{ij}\omega^i\omega^j \tag{5.1}$$

where the ω^i are the 1-forms defining the homogeneous Bianchi type I model. The g_{ij} are the metric functions, N and N_i respectively the lapse and shift functions. The relation between the proper time t and the time Ω is $dt^2 = (N^2 - N_iN^i)d\Omega^2$. In what follows we rewrite the metric functions as $g_{ij} = e^{-2\Omega+2\beta_{ij}}$ and use the Misner parameterisation[24] defined as:

$$\beta_{ij} = diag(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+) \tag{5.2}$$

$$p_k^i = 2\pi\pi_k^i - 2/3\pi\delta_k^i\pi_l^l \tag{5.3}$$

$$6p_{ij} = diag(p_+ + \sqrt{3}p_-, p_+ - \sqrt{3}p_-, -2p_+) \tag{5.4}$$

the p_{ij} being the conjugate momenta of β_{ij}. Hence the metric is cast into:

$$ds^2 = -(N^2 - N_iN^i)d\Omega^2 + 2N_i d\Omega\omega^i + R_0^2 e^{-2\Omega+2\beta_{ij}}\omega^i\omega^j \tag{5.5}$$

The most general form for the action is:

$$S = (16\pi)^{-1}\int [R - (3/2+\omega)\phi^{,\mu}\phi_{,\mu}\phi^{-2} - (3/2+\mu)\psi^{,\mu}\psi_{,\mu}\psi^{-2} - U \\ + 16\pi c^4 L_m]\sqrt{-g}d^4x \tag{5.6}$$

where a prime denotes ordinary derivation. ϕ and ψ are two scalar fields and their Brans-Dicke coupling functions with the metric are $\omega(\phi,\psi)$ and $\mu(\phi,\psi)$. $U(\phi,\psi)$ is the potential and L_m the Lagrangian of a

perfect fluid with an equation of state $p = (\gamma - 1)\rho$. We will consider the interval $\gamma \in [1,2]$ in which $\gamma = 1$ stands for a dust fluid, $\gamma = 4/3$ for a radiative fluid. Vacuum energy corresponds to $\gamma = 0$ and is equivalent to the presence of a cosmological constant which we will study. Defining the 3-volume V by $V = e^{-3\Omega}$ and using the energy impulsion conservation law for the perfect fluid, $T^{0\alpha}_{;\alpha} = 0$, we get for its energy density $\rho = V^{-\gamma}$.
Hamiltonian ADM formalism[78, 77] needs to rewrite the action under the following form:

$$S = (16\pi)^{-1}\int(\pi^{ij}\frac{\partial g_{ij}}{\partial t} + \pi^{\phi}\frac{\partial\phi}{\partial t} + \pi^{\psi}\frac{\partial\psi}{\partial t} - NC^0 - N_iC^i)d^4x \tag{5.7}$$

π_{ij}, π_ϕ and π_ψ are respectively the metric functions and scalar fields conjugate momenta. In the action (5.7), N and N_i play the role of Lagrange multipliers and C^0 and C^i are respectively the superhamiltonian and supermomenta. Considering (5.6) and (5.7), we deduce that:

$$\begin{aligned}
C^0 &= -\sqrt{^{(3)}g}\,^{(3)}R - \frac{1}{\sqrt{^{(3)}g}}(\frac{1}{2}(\pi^k_k)^2 - \pi^{ij}\pi_{ij}) + \frac{1}{2\sqrt{^{(3)}g}}(\frac{\pi^2_\phi\phi^2}{3+2\omega} + \frac{\pi^2_\psi\psi^2}{3+2\mu}) + \\
&\quad \sqrt{^{(3)}g}U + \frac{1}{\sqrt{^{(3)}g}}\frac{\delta e^{3(\gamma-2)\Omega}}{24\pi^2} \qquad (5.8)\\
C^i &= \pi^{ij}_{|j} \qquad (5.9)
\end{aligned}$$

where $|$ means covariant derivative on a $\{t = const\}$ surface. The variation of the action with respect to Lagrange multipliers leads to the constraints $C^0 = 0$ and $C^i = 0$. Using Misner parameterisation and the above definition of g_{ij}, we redefine the action (5.7) as $S = \int p_+ d\beta_+ + p_- d\beta_- + p_\phi d\phi + p_\psi d\psi - Hd\Omega$ with $p_\phi = \pi\pi_\phi$, $p_\psi = \pi\pi_\psi$ and $H = 2\pi\pi^k_k$, the ADM Hamiltonian. Then, the constraint $C^0 = 0$, yields for H:

$$H^2 = p^2_+ + p^2_- + 12\frac{p^2_\phi\phi^2}{3+2\omega} + 12\frac{p^2_\psi\psi^2}{3+2\mu} + 24\pi^2R^6_0e^{-6\Omega}U + \delta e^{3(\gamma-2)\Omega} \tag{5.10}$$

From (5.10), we derive the Hamiltonian equations:

$$\dot\beta_\pm = \frac{\partial H}{\partial p_\pm} = \frac{p_\pm}{H} \tag{5.11}$$

$$\dot\phi = \frac{\partial H}{\partial p_\phi} = \frac{12\phi^2p_\phi}{(3+2\omega)H} \tag{5.12}$$

$$\dot\psi = \frac{\partial H}{\partial p_\psi} = \frac{12\psi^2p_\psi}{(3+2\mu)H} \tag{5.13}$$

$$\dot p_\pm = -\frac{\partial H}{\partial\beta_\pm} = 0 \tag{5.14}$$

$$\begin{aligned}
\dot p_\phi &= -\frac{\partial H}{\partial\phi} = -12\frac{\phi p^2_\phi}{(3+2\omega)H} + 12\frac{\omega_\phi\phi^2p^2_\phi}{(3+2\omega)^2H} + 12\frac{\mu_\phi\psi^2p^2_\psi}{(3+2\mu)^2H} - \\
&\quad 12\pi^2R^6_0\frac{e^{-6\Omega}U_\phi}{H} \qquad (5.15)
\end{aligned}$$

$$\begin{aligned}
\dot p_\psi &= -\frac{\partial H}{\partial\psi} = -12\frac{\psi p^2_\psi}{(3+2\mu)H} + 12\frac{\omega_\psi\phi^2p^2_\phi}{(3+2\omega)^2H} + 12\frac{\mu_\psi\psi^2p^2_\psi}{(3+2\mu)^2H} - \\
&\quad 12\pi^2R^6_0\frac{e^{-6\Omega}U_\psi}{H} \qquad (5.16)
\end{aligned}$$

$$\dot H = \frac{dH}{d\Omega} = \frac{\partial H}{\partial\Omega} = -72\pi^2R^6_0\frac{e^{-6\Omega}U}{H} + 3/2\delta(\gamma-2)\frac{e^{3(\gamma-2)\Omega}}{H} \tag{5.17}$$

The dot means time derivative with respect to Ω. We choose the shift functions such that $N^i = 0$ and we find that the lapse function is related to the metric and the Hamiltonian by the relation $\partial\sqrt{g}/\partial\Omega = -1/2\pi^k_kN$ [78]. Thus, it comes:

$$N = \frac{12\pi R^3_0e^{-3\Omega}}{H} \tag{5.18}$$

The relation between Ω and the proper time t is then $dt = -Nd\Omega$. We wish to rewrite the Hamiltonian equations with normalised variables. The Hamiltonian (5.10), which also stands as a constraint equation, leads to the following choice:

$$x = H^{-1} \tag{5.19}$$

$$y = \sqrt{e^{-6\Omega}U}H^{-1} \tag{5.20}$$

$$z = p_\phi\phi(3+2\omega)^{-1/2}H^{-1} \tag{5.21}$$

$$w = p_\psi\psi(3+2\mu)^{-1/2}H^{-1} \tag{5.22}$$

It implies that $U > 0$, $3+2\omega > 0$ and $3+2\mu > 0$ so that the variables be real and the weak energy condition be satisfied. In addition, we define a variable depending on the above ones:

$$k^2 = \delta e^{3(\gamma-2)\Omega}H^{-2} = \delta y^2V^{-\gamma}U^{-1}$$

Some of these variables may be physically interpreted. x is proportional to the shear parameter Σ defined in [25] and k^2 to the density parameter Ω_m of the perfect fluid. We introduce them in the constraint (5.10) and get:

$$p^2x^2 + R^2y^2 + 12z^2 + 12w^2 + k^2 = 1 \tag{5.23}$$

with $p^2 = p_+^2 + p_-^2$ and $R^2 = 24\pi^2R_0^6$. This equation shows that the new variables (x,y,z,w) are normalised. Then, we rewrite the field equations as:

$$\dot{x} = 3R^2y^2x - 3/2(\gamma-2)k^2x \tag{5.24}$$

$$\dot{y} = y(6\ell_{\phi_1}z + 6\ell_{\psi_1}w + 3R^2y^2 - 3) - 3/2(\gamma-2)k^2y \tag{5.25}$$

$$\dot{z} = y^2R^2(3z - 1/2\ell_{\phi_1}) + 12w(w\ell_{\phi_2} - z\ell_{\psi_2}) - 3/2(\gamma-2)k^2z \tag{5.26}$$

$$\dot{w} = y^2R^2(3w - 1/2\ell_{\psi_1}) + 12z(z\ell_{\psi_2} - w\ell_{\phi_2}) - 3/2(\gamma-2)k^2w \tag{5.27}$$

where we have defined the folowing functions of the scalar fields ϕ and ψ

$$\begin{aligned}
\ell_{\phi_1} &= \phi U_\phi U^{-1}(3+2\omega)^{-1/2}\\
\ell_{\psi_1} &= \psi U_\psi U^{-1}(3+2\mu)^{-1/2}\\
\ell_{\phi_2} &= \phi\mu_\phi(3+2\mu)^{-1}(3+2\omega)^{-1/2}\\
\ell_{\psi_2} &= \psi\omega_\psi(3+2\omega)^{-1}(3+2\mu)^{-1/2}
\end{aligned}$$

Remark that these equations are unchanged under the transformation $x \leftrightarrow -x$ and/or $y \leftrightarrow -y$. Hence, we can limit our study to positive x or y. Moreover, some first degree equations for the scalar fields (which are not normalized) may be written as:

$$\dot{\phi} = 12z\frac{\phi}{\sqrt{3+2\omega}} \tag{5.28}$$

$$\dot{\psi} = 12w\frac{\psi}{\sqrt{3+2\mu}} \tag{5.29}$$

Hence, the nine Hamiltonian equations are rewritten with the six equations (5.24-5.29). This reduction in the number of equations comes from the fact that equations (5.14) imply $p_\pm \rightarrow consts$ and thus $\beta_+ \propto \beta_-$. Hence, it stays 9-3=6 equations to solve.

5.3 Assumptions

In this section we describe the assumptions we will use to study the above equations system. They concern the dependence of the Brans-Dicke functions and the potential with respect to the scalar fields, the type of equilibrium isotropic states we will consider and how fast it is approached by this system.

We will study the two **following classes of theories** from the Bianchi type I isotropisation viewpoint:

– For the first one, ω and μ will respectively depend on ϕ and ψ only, i.e. $\ell_{\phi_2} = \ell_{\psi_2} = 0$ whereas U will depend on both scalar fields. It means that the coupling between them only appears via the potential. As pointed in the introduction this type of theories may be obtained when one studies the hybrid inflation[114] or as the outcome of extra-dimensions compactification[113]. The theories of [114] and [113] are commented and studied from the isotropisation point of view in respectively the sections 5.6.1 and 5.6.2.

– For the second one, U and μ will depend on ψ only whereas ω will contain both scalar fields. Then, we will have $\ell_{\phi_1} = \ell_{\phi_2} = 0$. This type of theories is obtained when one casts a Lagrangian with one complex scalar field into another one with two real scalar fields. Complex scalar fields have been studied in [118] where the scalar fields quantization is considered, in [121] to study the formation of topological defects and in [122] for the Bose-Einstein condensate. We have analysed each theory of these papers from the isotropisation point of view in respectively the sections 5.6.3, 5.6.4 and 5.6.5.

Looking at the field equations, we have identified three types of isotropic equilibrium states (all characterised by $x \to 0$ when $\Omega \to -\infty$ as we will show it below) that we have classified into **three isotropisation classes**:

1. Class 1 is such as all the variables but not necessarily the scalar fields reach equilibrium with $y \neq 0$. Mathematically, it is the only one which allows to fully determine the asymptotical behaviours of the metric functions and potential in the vicinity of the isotropy.
2. Class 2 is such as all the variables but not necessarily the scalar fields reach equilibrium with $y = 0$. It is generally not possible to determine the asymptotical state of the system near isotropy because of y vanishing. If it is technically possible, the study of the general properties of this class will be the subject of future work.
3. Class 3 is such as at least $x_\pm$ reach equilibrium but not necessarily the other variables. If one of them behaves in this way, since it has to be bounded as $\Omega \to -\infty$, it would mean that it should be oscillating but not damped and then its first derivatives should oscillate around 0. It can never happen if one of the ℓ diverges monotonically or with sufficiently small oscillations since then, at least two of the derivatives $\dot{y}$, $\dot{z}$ or $\dot{w}$ will keep the same sign and thus will not be oscillating. It does not arise if the ℓ tend to some constants which is confirmed by numerical simulations. However, a partial equilibrium may occur for sufficiently oscillating ℓ which then allow an oscillation of the sign of the (y,z,w) first derivatives although $x_\pm$ tend to zero.

In this paper we will only study the first type of isotropic equilibrium state for the following reasons. *Mathematically* it is the only one allowing to determine completely the asymptotical behaviours of the metric functions and potential in the vicinity of the isotropy. *Physically*, near the isotropic state, either one of the scalar fields energy densities will be negligible with respect to the other or they will both behave in the same way. Let us assume without loss of generality that near the isotropic state the dominant scalar field energy density be the one of ϕ. y is then proportional to $(p_\phi - \rho_\phi)/Hubble^2$ where $Hubble$ is the Hubble function. Defining the scalar field parameter as $\Omega_\phi \propto \rho_\phi/Hubble^2$, the class 2 is thus such as $\Omega_\phi \to 0$ or $p_\phi \to \rho_\phi$ whereas the class 3 should be such as Ω_ϕ does not reach equilibrium. The class 1 is thus the only one such as asymptotically Ω_ϕ tends to a non vanishing constant. WMAP observations[182] indeed shows that today $\Omega_\phi = 0.73$ and $p_\phi/\rho_\phi < -0.78$.

As a last assumption, we will suppose that **the Universe approaches suffi ciently fastly its isotropic state**. This is a reasonable assumption since the Universe was already very isotropic at the CMB time and it will allow us to recover classical behaviours for the metric functions in the vicinity of the isotropic state, such as power and exponential laws of the proper time. All the asymptotical behaviours we will determine will be concerned by this assumption. Mathematically, it means that on one hand a function f of the scalar fields and on the other hand the variables (y,z,w,k) will have to tend sufficiently fastly to their equilibrium values such as their variations in the vicinity of the equilibrium may be neglected.
The form of the function $f(\phi,\psi)$ will be related on the presence or not of a perfect fluid and the dependence of the Brans-Dicke coupling functions and potential with respect to the scalar field. If in the neighbourhood of the equilibrium, f tends to a constant equilibrium value f_0, vanishing or not and such as $f \to f_0 + \delta f$ with $\delta f << f_0$, we will assume that $\int f d\Omega \to f_0\Omega + f_1$, f_1 being an integration constant. It will be equal to the constant f_1 if $f_0 = 0$. This assumption could be easily raised by keeping the integral but then our results will not be on a closed form and not easily physically interpretable. However it is mathematically feasible.
The same kind of assumptions will be made for the variables (y,z,w,k) with respect to $(\delta y,\delta z,\delta w,\delta k)$ but they can not be raised so easily. A perturbative analysis would be probably necessary and could depend on the particular form of the Brans-Dicke functions and potential with respect to the scalar fields whereas we wish to keep these functions undetermined.
The above assumptions are illustrated by an example in section 5.4.1 in the part "Asymptotic behaviours". In the section 5.6 where we will apply our results to some scalar-tensor theories, they will be systematically checked.

We will examine each of the above defined classes of scalar tensor theories, firstly without a perfect fluid

($k = 0$ strictly) and secondly with it ($k \neq 0$ or $k \to 0$). Equations (5.28-5.29) will serve to establish the scalar fields asymptotic behaviours.

5.4 Study of the equilibrium states

We are going to look for the equilibrium points representing an asymptotically isotropic Universe for the two classes of scalar tensor theories defined respectively by $\ell_{\phi_2} = \ell_{\psi_2} = 0$ and $\ell_{\phi_1} = \ell_{\phi_2} = 0$ and such as all the variables (x,y,z,w) reach equilibrium with $y \neq 0$.
In their famous paper [108], Collins and Hawking defined the isotropy as $\Omega \to -\infty$, in the following way

- Let $T_{\alpha\beta}$ be the energy-momemtum tensor: $T^{00} > 0$ and $\frac{T^{0i}}{T^{00}} \to 0$
  $\frac{T^{0i}}{T^{00}}$ represents a mean velocity of the matter compared to surfaces of homogeneity. If this quantity did not tend to zero, the Universe would not appear homogeneous and isotropic.
- Let be $\sigma_{ij} = (de^{\beta}/dt)_{k(i}(e^{-\beta})_{j)k}$ and $\sigma^2 = \sigma_{ij}\sigma^{ij}$: $\frac{\sigma}{d\Omega/dt} \to 0$, i.e. the shear parameter, proportional to x variables disappears. This condition says that the anisotropy measured locally through the Hubble parameter H_0 tends to zero.
- β tends to a constant β_0
  This condition is justified by the fact that the anisotropy measured in the CMB is to some extent a measurement of the change of the matrix β between time when radiation was emitted and time when it was observed. If β did not tend to a constant, one would expect large quantities of anisotropies in some directions.

Hence, in our calculations, we will look for the equilibrium states respecting the second point, i.e. isotropisation occurs when $x \to 0$ as $\Omega \to -\infty$. It is thus a stable state arising for a diverging value of t. These two limits do not depend on each other and their consistency will have to be checked. We will see that the third point will be always respected since $\beta_\pm$ will always disappear exponentially. The first point is also respected since we consider a diagonal tensor $T^{\alpha\beta}$ and positive energy densities for the perfect fluid and scalar field.
The system of first order equations (5.24-5.27) is not totally autonomous since ℓ_{ϕ_1}, ℓ_{ϕ_2}, ℓ_{ψ_1} and ℓ_{ψ_2} are some functions of ϕ and ψ. To make it fully autonomous, we have to consider the two additional first order equations (5.28-5.29) for ϕ and ψ. Since the scalar fields do not appear in the constraint, they do not need to be bounded whatever the isotropisation class. Hence, looking for stable isotropic states for class 1 isotropisation only consists in finding the values of (x,y,z,w) depending on the scalar fields such as $(\dot{x},\dot{y},\dot{z},\dot{w}) = (0,0,0,0)$. The equilibrium values of z and w will be introduced in equations (5.28) and (5.29) to respectively get ϕ and ψ asymptotic behaviours.

5.4.1 Without a perfect fluid

$\ell_{\phi_2} = \ell_{\psi_2} = 0$

In this subsection and the following ones, we first look for equilibrium points corresponding to isotropic stable states, i.e. such as $x = 0$. Then we search for monotonic functions and finally calculate the asymptotic behaviours of some important quantities in the neighbourhood of these points.

Calculus of the equilibrium points.
We find two equilibrium points:

$$(x,y,z,w) = (0, \pm\,(3 - \ell_{\phi_1}^2 - \ell_{\psi_1}^2)^{1/2}(\sqrt{3}R)^{-1}, \ell_{\phi_1}/6, \ell_{\psi_1}/6) \tag{5.30}$$

and a set of points defined by $y = 0$. The two first ones respect the constraint and are real if $\ell_{\phi_1}^2 + \ell_{\psi_1}^2$ tends to a constant smaller than 3. Both ℓ_{ϕ_1} and ℓ_{ψ_1} have to tend to a constant such as $\dot{z}$ and $\dot{w}$ vanish. We will show below that they are in agreement with a negatively diverging value of Ω. We do not look after the set of points defined by $y = 0$ since it concerns the class 2 isotropisation which we will not study in this work.

Monotonic functions.
By examining equation (5.24), we deduce that x is a monotonic function of Ω: if $x < 0$ ($x > 0$) initially, it will keep the same sign and will be decreasing (increasing). Thus, if initially the Hamiltonian is positive, from the definition (5.18) of the lapse function N and the relation between the proper time t and Ω, we derive that $\Omega \to -\infty$ corresponds to late times epoch. In the same way, if initially $z < \ell_{\phi_1}/6$ ($z > \ell_{\phi_1}/6$), z will be monotonically decreasing(increasing). The same conclusion arises for w related on the value $\ell_{\psi_1}/6$.

As in the case of a single scalar field[105], x being monotonic and with a constant sign, we are able to show that the metric functions may have one extremum at most. Indeed, their derivatives write as $dg_{ij}/dt = -2N^{-1}e^{-2\Omega+2\beta_{ij}}(\dot\beta_{ij}-1)$ and $\dot\beta_{ij}$ is a linear combination of $\dot\beta_\pm$ which depends on the monotonic function x. Consequently, there exists only one value for x such as dg_{ij}/dt vanishes. In a general way, if we consider the two Brans-Dicke coupling functions as depending on both scalar fields ϕ and ψ (i.e. $\ell_i \neq 0$ whatever $i = \phi_1,\phi_2,\psi_1,\psi_2$), z and w are not necessarily monotonic but it is always the case for x. Thus, whatever the dependence of ω, μ and U on ϕ and ψ, the metric functions will always have one extremum at most.
All these elements show that there is no periodic or homoclinic orbit in the phase space (x,y,z,w).

Asymptotic behaviours
As explained in the section 5.3, we define the function $f = \ell^2_{\phi_1} + \ell^2_{\psi_1}$ such as when $\Omega \to -\infty$, we assume that $\int f d\Omega \to (\ell^2_{\phi_1} + \ell^2_{\psi_1})\Omega + f_1$. Then, when Ω diverges and after having replaced y by its equilibrium value neglecting its variation δy in the vicinity of the equilibrium, we deduce from (5.24) that $x \to x_0 e^{\int (3-\ell^2_{\phi_1}-\ell^2_{\psi_1})d\Omega} \to x_0 e^{(3-\ell^2_{\phi_1}-\ell^2_{\psi_1})\Omega}$. It shows that the equilibrium points reality condition is in agreement with the vanishing of x when $\Omega \to -\infty$. Introducing this asymptotic expression for x in the lapse function (5.18), it is possible to calculate the asymptotic form of $e^{-\Omega}$ as a function of the proper time t, i.e. the metric functions attractor. Its local or global nature can not be determined unless we specify ω, μ or U. When $\ell^2_{\phi_1} + \ell^2_{\psi_1}$ tends to a non vanishing constant, $e^{-\Omega} \to t^{(\ell^2_{\phi_1}+\ell^2_{\psi_1})^{-1}}$. When it vanishes, $e^{-\Omega}$ tends to an exponential of t. We also evaluate the asymptotic forms of ϕ and ψ by rewriting the equations (5.28) and (5.29) near the equilibrium. We get two differential equations whose asymptotic solutions take the same forms as those of ϕ and ψ when $\Omega \to -\infty$:

$$\dot\phi = \frac{2\phi^2 U_\phi}{(3+2\omega)U} \tag{5.31}$$

$$\dot\psi = \frac{2\psi^2 U_\psi}{(3+2\mu)U} \tag{5.32}$$

Since $\dot U = U_\phi\dot\phi + U_\psi\dot\psi$ and applying our assumption on f, we determine that near equilibrium:

$$U \propto exp\left[2(\ell^2_{\phi_1} + \ell^2_{\psi_1})\Omega\right] \tag{5.33}$$

This shows that if $\ell^2_{\phi_1} + \ell^2_{\psi_1}$ tends to a non vanishing constant, U asymptotically vanishes as t^{-2}. If it vanishes, the potential tends to some constant. Note that the special case for which $\ell^2_{\phi_1} + \ell^2_{\psi_1} \to 3$ implies $y \to 0$ and thus belongs to class 2 isotropisation which will not be studied in this work. Approach of equilibrium is represented by a phase portrait diagram on figure 5.1.

$\ell_{\phi_1} = \ell_{\phi_2} = 0$

We proceed as in the previous subsection.

Calculus of the equilibrium points.
We find the following equilibrium points, E_1 and E_2, which might correspond to some isotropic stable states:

$$\begin{aligned}
E_1 &= (0, \pm(1-\ell^2_{\psi_1}/3)^{1/2}R^{-1}, 0, \ell_{\psi_1}/6) \\
E_2 &= (0, \pm\left[2\ell_{\psi_2}(\ell_{\psi_1}+2\ell_{\psi_2})^{-1}\right]^{1/2}R^{-1}, \\
&\quad \pm(\ell^2_{\psi_1}+2\ell_{\psi_1}\ell_{\psi_2}-3)^{1/2}\left[2\sqrt{3}(\ell_{\psi_1}+2\ell_{\psi_2})\right]^{-1}, \\
&\quad (2\ell_{\psi_1}+4\ell_{\psi_2})^{-1})
\end{aligned}$$

They both check the constraint equation. The first one will be real and bounded if $\ell^2_{\psi_1} \leq 3$ and tends to a constant. The second one needs that $\ell_{\psi_2}(\ell_{\psi_1}+2\ell_{\psi_2})^{-1}$ tends to a positive constant, $\ell_{\psi_1}(\ell_{\psi_1}+2\ell_{\psi_2}) \geq 3$ and $\ell_{\psi_1}+2\ell_{\psi_2} \neq 0$. Remark that for E_2, ℓ_{ψ_1} and ℓ_{ψ_2} may be unbounded. A third set of equilibrium points is $(y,z) = (0,0)$ but we discard it for the same reasons as in the previous subsection.

Monotonic functions.

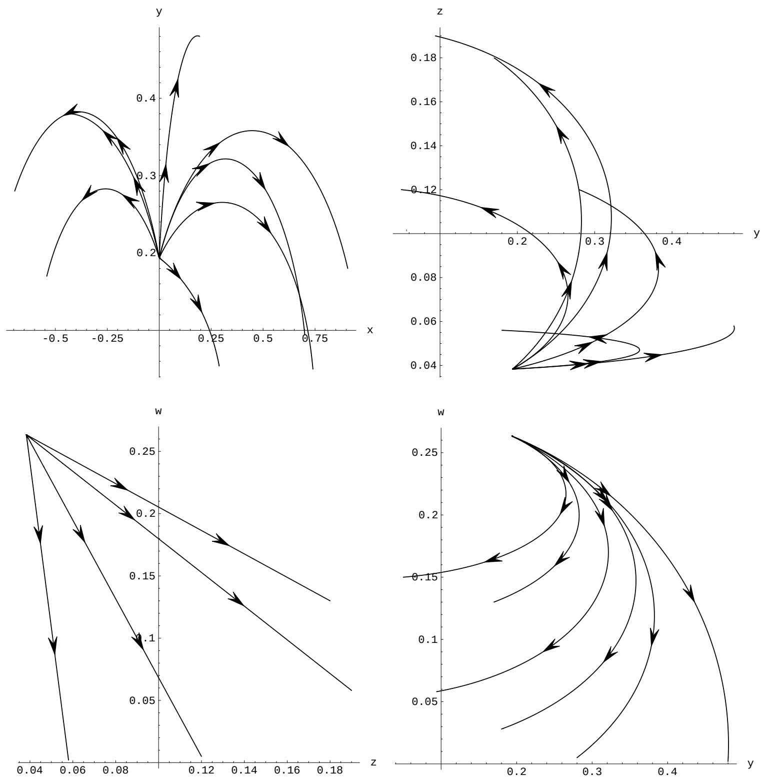

FIG. 5.1 – *Case 1A - Equilibrium point approach when no perfect fluid is present and* $(L_{\phi_1},L_{\phi_2},L_{\psi_1},L_{\psi_2},R,p) = (0.23,0,1.58,0,2,1)$. *The point is located at* $(x,y,z,w) = (0,0.19,0.04,0.26)$.

As written in subsection 5.4.1, x is a monotonic function of Ω and $\Omega(t)$ a monotonic function of the proper time whose limit $\Omega \to -\infty$ corresponds to late time epoch when the Hamiltonian is initially positive. If $\ell_{\psi_2} > 0$ ($\ell_{\psi_2} < 0$) and $w > \ell_{\psi_1}/6$ ($w < \ell_{\psi_1}/6$), w is an increasing (decreasing) function of Ω. If moreover $\ell_{\psi_1} > 0$ ($\ell_{\psi_1} < 0$), w is positive (negative) and keeps a constant sign.

Asymptotic behaviour in the neighbourhood of E_1

Here we define $f = \ell^2_{\psi_1}$ and write that in $\Omega \to -\infty$, $\int \ell^2_{\psi_1} d\Omega \to \ell^2_{\psi_1}\Omega + f_1$. Then, from (5.26) we get:

$$z \to e^{(3-\ell^2_{\psi_1})\Omega - 2\int \ell_{\psi_1}\ell_{\psi_2} d\Omega} \tag{5.34}$$

Indeed ℓ_{ψ_1} must tend to a constant but ℓ_{ψ_2} may diverge. It is why an integral of ℓ_{ψ_2} appears in this last expression. It shows that we must have $(3-\ell^2_{\psi_1})\Omega - 2\int \ell_{\psi_1}\ell_{\psi_2} d\Omega \to -\infty$ when $\Omega \to -\infty$ such as z vanishes. Moreover, considering equation (5.27) where a $z^2\ell_{\psi_2}$ term is present, we deduce that z has to vanish sufficiently fast to allow w equilibrium, i.e. $z^2\ell_{\psi_2} \to 0$. When the condition for z vanishing is respected, $\dot{z}z = (3-\ell^2_{\psi_1})z^2 - 2\ell_{\psi_1}\ell_{\psi_2}z^2 \to 0$ and we deduce that $z^2\ell_{\psi_2} \to 0$ is always true as long as (obviously) ℓ_{ψ_2} does not diverge or/and ℓ_{ψ_1} does not vanish. Otherwise, nothing can be deduced from $\dot{z}z$ vanishing.

The variable x behaves as $x_0 e^{(3-\ell^2_{\psi_1})\Omega}$ and vanishes as $\Omega \to -\infty$ when reality condition for the equilibrium points is respected. As previously, using the expression for the lapse function and the relation $dt = -Nd\Omega$, we get $e^{-\Omega}$ as a function of the proper time near isotropy. If ℓ_{ψ_1} tends to a non vanishing constant, $e^{-\Omega}$ tends to $t^{\ell^{-2}_{\psi_1}}$. If ℓ_{ψ_1} vanishes, $e^{-\Omega}$ tends to an exponential of the proper time. In the same way as in subsection 5.4.1, we calculate the differential equations whose solutions asymptotically correspond to the forms of ϕ and ψ when $\Omega \to -\infty$:

$$\dot{\phi} = 12\phi(3+2\omega)^{-1/2} e^{(3-\ell^2_{\psi_1})\Omega - 2\int \ell_{\psi_1}\ell_{\psi_2} d\Omega} \tag{5.35}$$

$$\dot{\psi} = \frac{2\psi^2 U_\psi}{(3+2\mu)U} \tag{5.36}$$

Since $\dot{U} = U_\psi\dot{\psi}$, it comes that:

$$U \propto e^{2\ell^2_{\psi_1}\Omega} \tag{5.37}$$

Thus, the potential behaves as t^{-2} when $\ell_{\psi_1}^2$ tends to a non vanishing constant or as a constant when $\ell_{\psi_1}^2$ tends to zero. Again if $\ell_{\psi_1}^2 \to 3$, y vanishes and thus isotropisation is of class 2. We thus exclude this value from our study.

Asymptotic behaviour in the neighbourhood of E_2

We define $f = \ell_{\psi_1}(\ell_{\psi_1} + 2\ell_{\psi_2})^{-1}$ and write that in $\Omega \to -\infty$, $\int \ell_{\psi_1}(\ell_{\psi_1} + 2\ell_{\psi_2})^{-1} d\Omega \to \ell_{\psi_1}(\ell_{\psi_1} + 2\ell_{\psi_2})^{-1}\Omega + f_1$. With this assumption we calculate that near E_2, x behaves as $x_0 e^{3\left[2\ell_{\psi_2}(\ell_{\psi_1}+2\ell_{\psi_2})^{-1}\right]\Omega}$. Since $2\ell_{\psi_2}(\ell_{\psi_1} + 2\ell_{\psi_2})^{-1}$ tends to a positive constant, it thus vanishes as $\Omega \to -\infty$. Using the expression (5.18) for the lapse function, we calculate that when the quantity $1 - 2\ell_{\psi_2}(\ell_{\psi_1} + 2\ell_{\psi_2})^{-1} = \ell_{\psi_1}(\ell_{\psi_1} + 2\ell_{\psi_2})^{-1}$ tends to a non vanishing constant, $e^{-\Omega} \to t^{(\ell_{\psi_1}+2\ell_{\psi_2})(3\ell_{\psi_1})^{-1}}$. From reality condition for the point E_2, we deduce that this power of t is positive. Hence, this last expression is increasing when the proper time t diverges in accordance with the growth of $e^{-\Omega}$ when $\Omega \to -\infty$. If the quantity $\ell_{\psi_1}(\ell_{\psi_1} + 2\ell_{\psi_2})^{-1}$ vanishes, the metric functions tend to an exponential of the proper time. From (5.12) and (5.13) we derive the differential equations allowing ϕ and ψ to get asymptotic forms:

$$\dot{\phi} = -2\sqrt{3}\frac{\phi}{\psi}\frac{\sqrt{-3U^2(3+2\mu)(3+2\omega) + \psi^2 U_\psi \left[U(3+2\omega)\right]_\psi}}{\left[U(3+2\omega)\right]_\psi} \tag{5.38}$$

$$\dot{\psi} = \frac{6U(3+2\omega)}{\left[U(3+2\omega)\right]_\psi} \tag{5.39}$$

This last equation easily integrates to give $U(3+2\omega) = e^{6(\Omega-\Omega_0)}$, Ω_0 being an integration constant. Since we have $\dot{U} = U_\psi\dot{\psi}$, we deduce from (5.39) that $U = e^{6\ell_{\psi_1}(\ell_{\psi_1}+2\ell_{\psi_2})^{-1}\Omega}$. Consequently, when $\ell_{\psi_1}(\ell_{\psi_1} + 2\ell_{\psi_2})^{-1}$ tends to a non vanishing constant, the potential vanishes as t^{-2}. If it vanishes, the potential tends to a non vanishing constant.

Approaches of both equilibrium points are represented by phase portrait diagrams on figures 5.2 and 5.3.

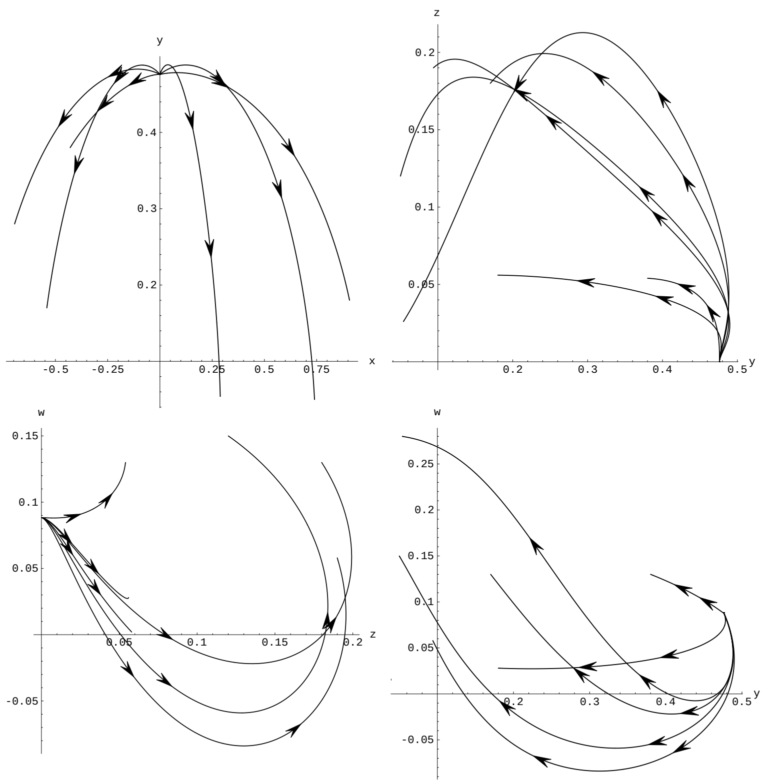

FIG. 5.2 – *Case 2A - First equilibrium point approach when no perfect fluid is present and* $(L_{\phi_1},L_{\phi_2},L_{\psi_1},L_{\psi_2},R,p) = (0,0,0.53,1,2,1)$. *The point is located at* $(x,y,z,w) = (0,0.47,0,0.09)$.

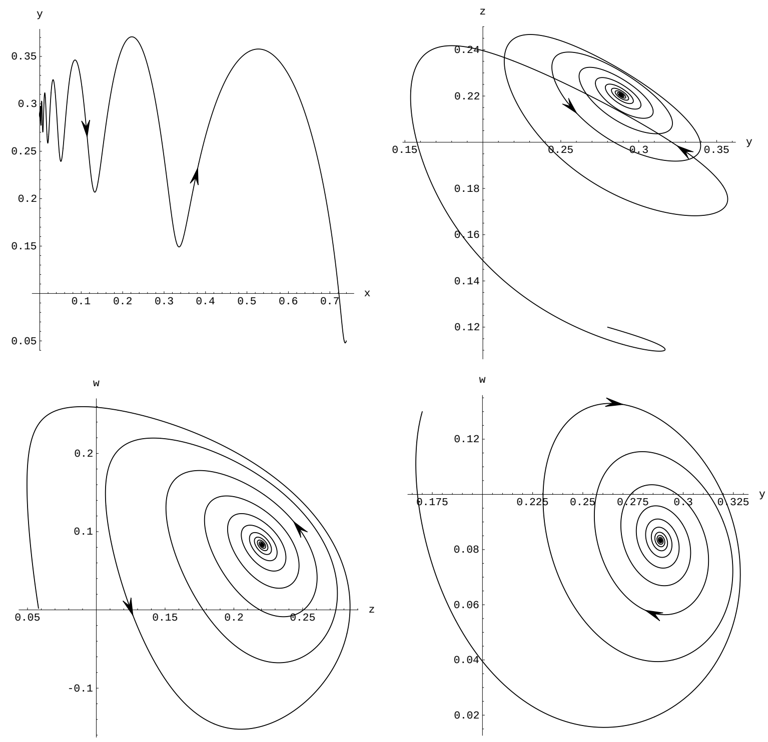

FIG. 5.3 – *Case 2A - Second equilibrium point approach when no perfect fluid is present and* $(L_{\phi_1},L_{\phi_2},L_{\psi_1},L_{\psi_2},R,p) = (0,0,4,1,2,1)$. *Let us note how this approach is different from the first equilibrium point.* x *and* y *undergo damped oscillations when they approach their equilibrium values. The point is located at* $(x,y,z,w) = (0,0.29,0.22,0.08)$.

5.4.2 With a perfect fluid

There are two types of equilibrium points when we take into account a perfect fluid depending if k, or equivalently the density parameter of the perfect fluid, tends to a non vanishing or vanishing constant. The first type is studied in the two next subsections and the second one in the third subsection.

$\ell_{\phi_2} = \ell_{\psi_2} = 0$

Calculus of the equilibrium points.
In the annexe 2, we look for the zeros of (5.24-5.27) and introduce them in the constraint to determine k. The only ones in agreement with isotropy are:

$$E_{4,5} = (0, \pm 1/2\sqrt{3}R^{-1}\left[\gamma(2-\gamma)(\ell_{\phi 1}^2+\ell_{\psi 1}^2)^{-1}\right]^{1/2}, 1/4\gamma\ell_{\phi 1}(\ell_{\phi 1}^2+\ell_{\psi 1}^2)^{-1},$$
$$1/4\gamma\ell_{\psi 1}(\ell_{\phi 1}^2+\ell_{\psi 1}^2)^{-1})$$

with $k^2 \to 1 - \frac{3\gamma}{2(\ell_{\phi 1}^2+\ell_{\psi 1}^2)}$ that is real and non vanishing if $\ell_{\phi 1}^2+\ell_{\psi 1}^2 > 3/2\gamma$. We will show in this subsection that k tends to non vanishing constant and it is why we exclude the special value $\ell_{\phi 1}^2+\ell_{\psi 1}^2 \to 3/2\gamma$. Since we study the class 1 isotropisation, y is non vanishing and then $\ell_{\phi 1}^2+\ell_{\psi 1}^2$ can not diverge. Hence, from the forms of $E_{4,5}$ points, we deduce that respectively the sum $\ell_{\phi 1}^2+\ell_{\psi 1}^2$ and its individual components have to tend to some constants.

Monotonic functions
Equation (5.24) shows that x is a monotonic function with a constant sign whatever the values of ℓ_{ϕ_2}, ℓ_{ψ_2}, ℓ_{ϕ_1} and ℓ_{ψ_1}. Consequently the lapse function also has a constant sign and Ω is a monotonically decreasing function of t if initially $H > 0$, tending to $-\infty$ for late times. If $z > \ell_{\phi 1}$, z is a monotonically increasing function. However, nothing can be deduced when $z < \ell_{\phi 1}$ because of the perfect fluid presence. The same reasoning holds for w with respect to $\ell_{\psi 1}$. Hence, it seems that no periodic or homoclinic orbit exists.

Asymptotic behaviours

Here, there is no need to make any assumptions related to a function f as defined in the subsection 5.3. Using (5.28-5.29), the scalar fields asymptotic behaviours are defined by the asymptotic solutions of the following systems:

$$\dot{\phi} = 3\gamma \frac{(3+2\mu)\phi^2 U U_\phi}{(3+2\mu)\phi^2 U_\phi^2 + (3+2\omega)\psi^2 U_\psi^2} \tag{5.40}$$

$$\dot{\psi} = 3\gamma \frac{(3+2\omega)\phi\psi U U_\psi}{(3+2\mu)\phi^2 U_\phi^2 + (3+2\omega)\psi^2 U_\psi^2} \tag{5.41}$$

Linearising (5.24) in the neighbourhood of $E_{4,5}$, we find that asymptotically $x \to e^{-\frac{3}{2}(\gamma-2)\Omega}$ and vanishes as $\Omega \to -\infty$ for the considered interval of γ. Then, from the lapse function N, it comes that $e^{-\Omega} \to t^{\frac{2}{3\gamma}}$. We deduce from y definition and the fact that this variable does not vanish near equilibrium that $U \to t^{-2} \propto V^{-\gamma}$. In the same way we deduce from k definition, using the asymptotic expressions for $U(t)$ and $\Omega(t)$, that it tends to a non vanishing constant. It is thus the same for the density parameter Ω_m of the perfect fluid. Approach to equilibrium is represented by the phase portrait diagram on figure 5.4.

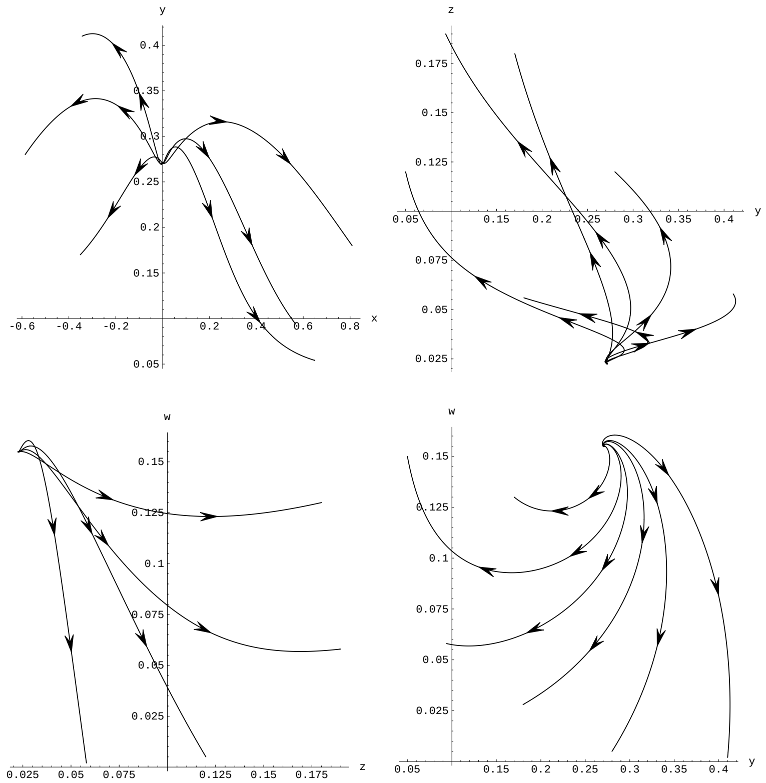

FIG. 5.4 – *Case 1B - Equilibrium point approach when a perfect fluid with* $\gamma = 1$ *is present and* $(L_{\phi_1}, L_{\phi_2}, L_{\psi_1}, L_{\psi_2}, R, p, k) = (0.23, 0, 1.58, 0, 2, 1, 0.41)$. *The point is located at* $(x, y, z, w) = (0, 0.27, 0.022, 0.15)$.

$\ell_{\phi_1} = \ell_{\phi_2} = 0$

Calculus of the equilibrium points.

We proceed as in the previous section. The details for the equilibrium points calculus are given in the annexe 2. We find that the only ones corresponding to an isotropic state are:

$$E_{2,3} = (0, \pm 1/2 R^{-1} \ell_{\psi 1}^{-1} \sqrt{3\gamma(2-\gamma)}, 0, 1/4\gamma \ell_{\psi 1}^{-1}) \tag{5.42}$$

with $k^2 \to 1 - 3/2\gamma \ell_{\psi_1}^{-2}$ and thus require $\ell_{\psi 1}^2 > 3/2\gamma$. We will show below that k is non vanishing and it is why we exclude the value $\ell_{\psi 1}^2 \to 3/2\gamma$. Moreover, since we consider a class 1 isotropisation, y can not tend to zero and $\ell_{\psi 1}$ is bounded. Hence, equilibrium is reached only if $\ell_{\psi 1}$ tends to a constant such as $\dot{y}$ and

$\dot{z}$ vanish.

Monotonic functions
As already noted in the previous section, x is a monotonic function of Ω and has a constant sign. Consequently, Ω is a monotonic decreasing function of t if initially the Hamiltonian is positive and $\Omega \to -\infty$ corresponds to late times epochs.

Asymptotic behaviours
Once again, there is no need to make any assumptions related to a function f as defined in the subsection 5.3. Surprisingly, equilibrium points $E_{2,3}$ have the same form as in the presence of a single scalar field ψ in [109]. If we consider a Lagrangian with a single complex scalar field ζ and cast it into another Lagrangian with 2 real scalar fields ϕ and ψ, $E_{2,3}$ would only depend on its amplitude ψ and not on its phase ϕ.
Again, we find that asymptotically $x \to e^{-\frac{3}{2}(\gamma-2)\Omega}$ and thus $e^{-\Omega} \to t^{\frac{2}{3\gamma}}$ independently on the scalar fields behaviours. Hence considering the definition of y, the potential vanishes as $t^{-2} \propto V^{-\gamma}$. As in the previous section, one can show using these asymptotical behaviours for the metric functions and potential that k tends to a non vanishing constant. The differential equation giving ψ in $\Omega \to -\infty$ may be written:

$$\dot{\psi} = 3\gamma \frac{U}{U_\psi} \tag{5.43}$$

To determine a similar equation for ϕ, we need to know z asymptotic behaviour. We find $z \to e^{3\left[(1-\gamma/2)\Omega - \gamma\int \ell_{\psi 2}\ell_{\psi 1}^{-1} d\Omega\right]}$ and it is thus vanishing when $\Omega \to -\infty$ if $(1-\gamma/2)\Omega - \gamma\int \ell_{\psi 2}\ell_{\psi 1}^{-1} d\Omega \to -\infty$. Then, the differential equation giving the asymptotic form for ϕ is:

$$\dot{\phi} = \phi_0 \frac{12\phi}{\sqrt{3+2\omega}} e^{3\left[(1-\gamma/2)\Omega - \gamma\int \ell_{\psi 2}\ell_{\psi 1}^{-1} d\Omega\right]} \tag{5.44}$$

ϕ_0 being an integration constant. As in the absence of a perfect fluid, z has to vanish sufficiently fast such that w reaches equilibrium, i.e. we must have $z^2\ell_{\psi 2} \to 0$. This condition is always satisfied as long as the one allowing the vanishing of z is respected, since we have then $\dot{z}z = 3(1-\gamma/2)z^2 - 3\gamma\ell_{\psi 2}\ell_{\psi 1}^{-1}z^2 \to 0$ and $\ell_{\psi 1}$ does not diverge. Approach to equilibrium is represented by the phase portrait diagram on figure 5.5.

The case $k \to 0$

As shown above, the limit $k \to 0$ disagrees with the isotropic equilibrium states defined for the equilibrium points of subsections 5.4.2-5.4.2. However, it is always possible to assume $k \to 0$ in the field equations and then to solve them. We thus recover the equilibrium points obtained in the absence of a perfect fluid. The asymptotic behaviours of x and of the metric functions are the same as in section 5.4.1. However, the conditions for isotropisation are modified since now k has to vanish asymptotically, thus representing an additional constraint. To find it, we rewrite k as $\delta x^2 e^{(3\gamma-6)\Omega}$. Then, in the case $\ell_{\phi_2} = \ell_{\psi_2} = 0$ where $x \to e^{3-\ell_{\phi_1}^2-\ell_{\psi_1}^2}$, k will vanish only if $\ell_{\phi_1}^2 + \ell_{\psi_1}^2 < 3/2\gamma$ which is consistent, although more restrictive, with reality condition of the equilibrium points when no perfect fluid is present since $\gamma \in$ [1,2]. Hence, $\ell_{\phi_1}^2 + \ell_{\psi_1}^2 > 3/2\gamma$ is a necessary condition for isotropisation to occur with $k \neq 0$ toward the equilibrium points of subsection 5.4.2, whereas $\ell_{\phi_1}^2 + \ell_{\psi_1}^2 < 3/2\gamma$ is a necessary condition for isotropisation to occur with $k \to 0$ toward the equilibrium points of subsection 5.4.1. The same reasoning may be followed concerning the case $\ell_{\phi_1} = \ell_{\phi_2} = 0$. For the E_1 point, k vanishes only if $\ell_{\psi_1}^2 < 3/2\gamma$ and for the E_2 point, if $2\ell_{\psi_2}(\ell_{\psi_1+2\ell_{\psi_2}})^{-1} > 1-\gamma/2$ with $1-\gamma/2 \in$ [0,1/2].
Since $k = \delta y^2 U^{-1} V^{-\gamma}$ and we consider the class 1 isotropisation such as $y \neq 0$, k vanishing implies $U >> V^{-\gamma}$ and thus, in the Lagrangian field equations, the potential will dominate the perfect fluid energy density term.

The results of this last subsection are more accurate and extended than those we had found in [109]. In this last paper, we had considered different cases depending on the behaviour of U with respect to $V^{-\gamma}$ and then deduced this of k. In the present paper, the opposite reasoning is made and seems to give better results. In particular in [109], we had not detected that conditions for isotropisation were changed when $k \to 0$ with respect to the no perfect fluid case.

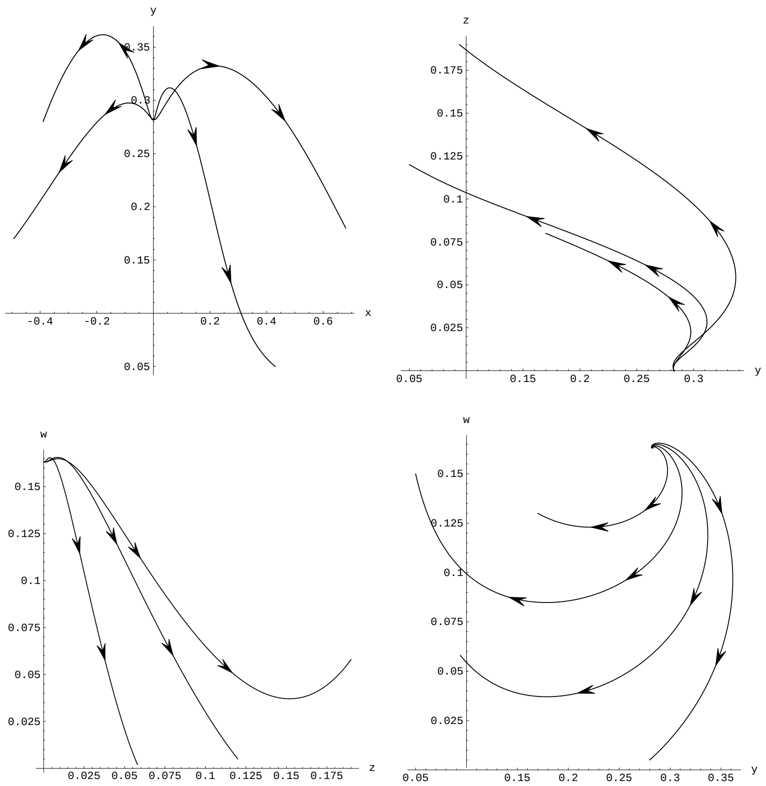

FIG. 5.5 – *Case 2B - Equilibrium point approach when a perfect fluid is present and* $(L_{\phi_1},L_{\phi_2},L_{\psi_1},L_{\psi_2},R,p,k) = (0,0,1.53,0.23,2,1,0.60)$. *The point is located at* $(x,y,z,w) = (0,0.28,0,0.16)$.

5.4.3 Technical results summary

In this subsection, we summarise our technical results. We got the hamiltonian field equations for a Bianchi type I Universe filled with a perfect fluid and two scalar fields defined by $\ell_{\phi_2} = \ell_{\psi_2} = 0$ and $\ell_{\phi_1} = \ell_{\phi_2} = 0$. We rewrote them with normalised variables and looked for the stable isotropic states defined such as the shear disappears when the Universe expands, i.e $x \to 0$ when $\Omega \to -\infty$. We then found the equilibrium points summarising in table 5.1 and depending on the asymptotic behaviour of k or equivalently the perfect fluid density parameter Ω_m.

	$\ell_{\phi_2} = \ell_{\psi_2} = 0$	$\ell_{\phi_1} = \ell_{\phi_2} = 0$
$\Omega_m = 0$ or $\Omega_m \to 0$	$(0,\frac{\pm(3-\ell_{\phi_1}^2-\ell_{\psi_1}^2)^{1/2}}{\sqrt{3}R},\frac{\ell_{\phi_1}}{6},\frac{\ell_{\psi_1}}{6})$	$E_1 = (0,\frac{\pm(1-\ell_{\psi_1}^2/3)^{1/2}}{R},0,\frac{\ell_{\psi_1}}{6})$ $E_2 = (0,\pm\frac{\left[2\ell_{\psi_2}(\ell_{\psi_1}+2\ell_{\psi_2})^{-1}\right]^{1/2}}{R},$ $\pm\frac{(\ell_{\psi_1}^2+2\ell_{\psi_1}\ell_{\psi_2}-3)^{1/2}}{2\sqrt{3}(\ell_{\psi_1}+2\ell_{\psi_2})},\frac{1}{(2\ell_{\psi_1}+4\ell_{\psi_2})})$
$\Omega_m \to const$ with $const \neq 0$	$(0,\pm\frac{1/2\sqrt{3}}{R}\left[\gamma(2-\gamma)(\ell_{\phi 1}^2+\ell_{\psi 1}^2)^{-1}\right]^{1/2},$ $\frac{\gamma\ell_{\phi 1}}{4(\ell_{\phi 1}^2+\ell_{\psi 1}^2)},\frac{\gamma\ell_{\psi 1}}{4(\ell_{\phi 1}^2+\ell_{\psi 1}^2)})$	$(0,\pm\frac{1}{2R\ell_{\psi 1}}\sqrt{3\gamma(2-\gamma)},0,\frac{\gamma}{4\ell_{\psi 1}})$

TAB. 5.1 – *The* (x,y,z,w) *equilibrium points representing stable isotropic states for the Universe*

We then found some asymptotical necessary conditions for isotropy depending on inequality and limits written with respect to the functions ℓ of the scalar fields. From the viewpoint of asymptotical behaviours, it comes asymptotically that either the potential vanishes as t^2 or tends to a constant. In this last case, the Universe tends to a De Sitter one whereas when Ω_m tends to a non vanishing constant, the metric functions behave as $t^{\frac{2}{3\gamma}}$ as in the absence of any scalar fields. In the other cases, they behave as some powers of the proper time, the powers beeing some constants defined as asymptotical limits of some scalar fields functions summarised in table 5.2.

$\ell_{\phi_2}=\ell_{\psi_2}=0$	$\ell_{\phi_1}=\ell_{\phi_2}=0$
$\ell^2_{\phi_1}+\ell^2_{\psi_1}$	$E_1:\ell^2_{\psi_1}$ $E_2:(\ell_{\psi_1}+2\ell_{\psi_2})(3\ell_{\psi_1})^{-1}$

TAB. 5.2 – *When the potential desappears and Ω_m does not tend to a constant, the metric functions behave as some power laws of the proper time in the neighbourhood of the isotropic state. These powers are summarised in this table for the corresponding isotropic equilibrium point.*

5.5 Discussion

After the tedious computation of previous sections, we now summarize and discuss our results. We have classified isotropisation process into three classes and looked for necessary conditions allowing for class 1 isotropisation of Bianchi type I model when two minimally and massive scalar fields with a perfect fluid are considered. The class 1 is such as all the variables (x,y,z,w) reach equilibrium with $y\neq 0$. We have assumed that the potential is positive, the scalar field respects the weak energy condition and the Universe isotropises sufficiently fastly such as we could neglect the variation of $f(\phi,\psi)$ and (y,z,w,k) in the vicinity of the equilibrium.
The first necessary conditions we have found for isotropisation stem from the definition of isotropy: it will only happen for a vanishing shear ($x\rightarrow 0$) and an eternally expanding Universe ($\Omega\rightarrow-\infty$), thus implying that an isotropic state is always stable. Moreover, it will correspond to late time isotropisation if the Hamiltonian is initially positive. Two classes of theories have been examined depending on the relation between (ω,μ,U) and (ϕ,ψ). Each of them has been studied without or with a perfect fluid.

Case A: without a perfect fluid

When the perfect fluid is not present, we have for class 1 isotropisation:

Case 1A: $\omega(\phi)$, $\mu(\psi)$ and $U(\phi,\psi)$
A necessary condition for isotropisation of Bianchi type I model when two minimally and massive scalar fields are present will be that the two quantities $\ell_{\phi_1}=\phi U_\phi U^{-1}(3+2\omega)^{-1/2}$ and $\ell_{\psi_1}=\psi U_\psi U^{-1}(3+2\mu)^{-1/2}$ tend to some constants such as $\ell^2_{\phi_1}+\ell^2_{\psi_1}<3$. When isotropisation occurs and one of the two constants is non vanishing, the power law $t^{(\ell^2_{\phi_1}+\ell^2_{\psi_1})^{-1}}$ is a late time attractor of the metric functions and the potential vanishes as t^{-2}. If the two constants vanish, the de Sitter Universe represents the late time attractor and the potential tends to a constant.

If we put $\ell_{\psi_1}=0$ strictly, we recover the results got in presence of a single scalar field without a perfect fluid [105]. Results of case 1A can be generalised for n scalar fields ϕ_i when their associated Brans-Dicke coupling functions ω_i respectively depend on only ϕ_i (see annexe 1). For that, it is sufficient to replace $\ell^2_{\phi_1}+\ell^2_{\psi_1}$ by $\sum_i\ell_i^2$. In the literature, it has been shown that the presence of multiple scalar fields could help to generate inflation: it is the **assisted inflation**[117]. Sometimes it is the opposite which happens: *the more scalar fields there are, the less likely the inflation occurs[117]. It seems that it is this last behaviour which arises for case $1A$: the more scalar fields there are, the more they contribute to the denominator of the power law to which the metric functions converge, and the less likely it will be larger than 1 and produce an accelerated expansion at late times*.
To evaluate the above conditions for a specific theory, we need to know the asymptotic behaviours for ϕ and ψ such as we could calculate ℓ_{ψ_1} and ℓ_{ψ_2}. It comes:

asymptotic behaviours of the scalar fields for case 1A
The asymptotic behaviours of the two scalar fields when an isotropic state is reached are those of the two functions ϕ and ψ when $\Omega\rightarrow-\infty$ defined by:

$$\dot\phi=\frac{2\phi^2U_\phi}{(3+2\omega)U} \tag{5.45}$$

$$\dot\psi=\frac{2\psi^2U_\psi}{(3+2\mu)U} \tag{5.46}$$

Now we summarize our results concerning the second type of coupling for ω, μ and U:

Case 2A: $\omega(\phi,\psi)$, $\mu(\psi)$ and $U(\psi)$

There exists two equilibrium points E_1 and E_2 which may correspond to an isotropic equilibrium state with two minimally and massive scalar fields for the Bianchi type I model. The necessary conditions to reach equilibrium are expressed with the two quantities $\ell_{\psi_1} = \psi U_\psi U^{-1}(3+2\mu)^{-1/2}$ and $\ell_{\psi_2} = \psi\omega_\psi(3+2\omega)^{-1}(3+2\mu)^{-1/2}$:

- *For point E_1, it is necessary that $\ell_{\psi_1}^2 < 3$ and $(3-\ell_{\psi_1}^2)\Omega - 2\int \ell_{\psi_1}\ell_{\psi_2}d\Omega \to -\infty$. When isotropisation occurs and if ℓ_{ψ_1} tends to a non vanishing constant, $t^{\ell_{\psi_1}^{-2}}$ is the late time attractor of the metric functions and the potential vanishes as t^{-2}. If ℓ_{ψ_1} tends to zero, a de Sitter Universe is the late time attractor and the potential tends to a constant. If moreover ℓ_{ψ_2} diverges, an additional condition for isotropisation is $\ell_{\psi_2}e^{2\left[(3-\ell_{\psi_1}^2)\Omega-2\int \ell_{\psi_1}\ell_{\psi_2}d\Omega\right]} \to 0$.*
- *For point E_2, it is necessary that $0 < 2\ell_{\psi_2}(\ell_{\psi_1}+2\ell_{\psi_2})^{-1} < 1$, $\ell_{\psi_1}+2\ell_{\psi_2} \neq 0$ and $\ell_{\psi_1}(\ell_{\psi_1}+2\ell_{\psi_2}) > 3$. When isotropisation occurs and if $\ell_{\psi_1}(\ell_{\psi_1}+2\ell_{\psi_2})^{-1}$ tends to a non vanishing constant, the late time attractor of the metric functions is a power law of the proper time $t^{(\ell_{\psi_1}+2\ell_{\psi_2})(3\ell_{\psi_1})^{-1}}$ and the potential vanishes as t^{-2}. If $\ell_{\psi_1}(\ell_{\psi_1}+2\ell_{\psi_2})^{-1}$ vanishes, a de Sitter Universe is the late time attractor and the potential tends to a constant.*

Contrary to what happens with only a single scalar field, there are now two equilibirum points. The first one is such as the metric functions asymptotic behaviour only depends on ψ whereas for the second one, it depends on both scalar fields ϕ and ψ, the power law exponent being totally different from the previous case. The scalar fields asymptotic behaviours allowing to calculate the quantities ℓ_{ψ_1} and ℓ_{ψ_2} are given in the following way:

asymptotic behaviours of the scalar fields for case 2A

The asymptotic behaviours of the two scalar fields when an isotropic state is reached are those of the two functions ϕ and ψ as $\Omega \to -\infty$ defined by:

- *for the equilibrium point E_1:*

$$\dot{\phi} = 12\phi(3+2\omega)^{-1/2}e^{(3-\ell_{\psi_1}^2)\Omega-2\int \ell_{\psi_1}\ell_{\psi_2}d\Omega} \tag{5.47}$$

$$\dot{\psi} = \frac{2\psi^2 U_\psi}{(3+2\mu)U} \tag{5.48}$$

- *for the equilibrium point E_2:*

$$\dot{\phi} = -2\sqrt{3}\frac{\phi}{\psi}\frac{\sqrt{-3U^2(3+2\mu)(3+2\omega)+\psi^2 U_\psi\left[U(3+2\omega)\right]_\psi}}{\left[U(3+2\omega)\right]_\psi} \tag{5.49}$$

$$U(3+2\omega) = e^{6(\Omega-\Omega_0)} \tag{5.50}$$

Let us examine the connection between our results and **Wald's No Hair theorem**[49]. The latter states that initially expanding homogeneous models with a positive cosmological constant (except Bianchi type IX) and a stress energy tensor satisfying the dominant and strong energy conditions, exponentially evolve to an isotropic de Sitter solution. The behaviour of Bianchi type IX model is similar if the cosmological constant is sufficiently large compared with spatial-curvature terms. Assuming that the Universe tends sufficiently fastly to its isotropic equilibrium state [1], Wald's No Hair theorem can be generalised for the case 1A to any form of potential and Brans-Dicke coupling functions such as ℓ_{ϕ_1} and ℓ_{ψ_1} tend to zero. For the case 2A and the equilibrium point E_1, only the vanishing of ℓ_{ψ_1} is necessary and, for the equilibrium point E_2, the vanishing of $\ell_{\psi_1}(\ell_{\psi_1}+2\ell_{\psi_2})^{-1}$. For both cases, the potential tends to a constant, showing the stability of Wald theorem with respect to the presence of several scalar fields. Note that the relations between Bianchi models and Wald's No Hair theorem has been explored in the context of chaotic inflation in [189].

Case B: with a perfect fluid

When we take into account a perfect fluid, we get different conditions and metric functions asymptotic behaviours resulting from class 1 isotropisation. There exists two possible equilibrium points respectively corresponding to a vanishing or non vanishing k or equivalently the perfect fluid density parameter Ω_m. For the first case, we have:

Case 1B: $\omega(\phi)$, $\mu(\psi)$ and $U(\phi,\psi)$.

1. i.e. considering that the assumptions on $f(\phi,\psi)$ and (y,z,w,k) we explained in the section 5.3 are checked

A necessary condition for isotropisation of Bianchi type I model when 2 massive and minimally coupled scalar fields with a perfect fluid are considered and such as $\Omega_m \to const \neq 0$ will be that the quantities $\ell_{\phi_1} = \phi U_\phi U^{-1}(3+2\omega)^{-1/2}$ and $\ell_{\psi_1} = \psi U_\psi U^{-1}(3+2\mu)^{-1/2}$ tend to some constants with $\ell_{\phi_1}^2 + \ell_{\psi_1}^2 > 3/2\gamma$. Then, when isotropisation occurs, the metric functions will always tend to $t^{\frac{2}{3\gamma}}$ and the potential will vanish as t^{-2}. When isotropisation arises such as $\Omega_m \to 0$, we recover the results of case 1A (including the scalar fields asymptotic behaviours) but the condition on $\ell_{\phi_1}^2 + \ell_{\psi_1}^2$ is cast into $\ell_{\phi_1}^2 + \ell_{\psi_1}^2 < 3/2\gamma$.

Hence, when $\Omega_m \to const \neq 0$, the metric functions asymptotic behaviour is the same as in presence of a single scalar field[109]. We find for the scalar fields asymptotic behaviours:

Asymptotic behaviours of the scalar fields for case 1B ($k \neq 0$)
The asymptotic behaviours of the two scalar fields when an isotropic state is reached with $\Omega_m \to const \neq 0$ are those of the two functions ϕ and ψ as $\Omega \to -\infty$ defined by:

$$\dot{\phi} = 3\gamma \frac{(3+2\mu)\phi^2 U U_\phi}{(3+2\mu)\phi^2 U_\phi^2 + (3+2\omega)\psi^2 U_\psi^2} \tag{5.51}$$

$$\dot{\psi} = 3\gamma \frac{(3+2\omega)\phi\psi U U_\psi}{(3+2\mu)\phi^2 U_\phi^2 + (3+2\omega)\psi^2 U_\psi^2} \tag{5.52}$$

If we consider the second type of coupling, we get the following results:

Case 2B: $\omega(\phi,\psi)$, $\mu(\psi)$ and $U(\psi)$.
Let be the quantities $\ell_{\psi_1} = \psi U_\psi U^{-1}(3+2\mu)^{-1/2}$ and $\ell_{\psi_2} = \psi\omega_\psi(3+2\omega)^{-1}(3+2\mu)^{-1/2}$. Some necessary conditions for isotropisation of Bianchi type I model when 2 massive and minimally coupled scalar fields with a perfect fluid are considered and such as $\Omega_m \to const \neq 0$ will be that ℓ_{ψ_1} tends to a constant with $\ell_{\psi_1}^2 > 3/2\gamma$ and $(1-\gamma/2)\Omega - \gamma\int \ell_{\psi 2}\ell_{\psi 1}^{-1} d\Omega \to -\infty$ as $\Omega \to -\infty$. When isotropisation arises, the metric functions will always tend to $t^{\frac{2}{3\gamma}}$ and the potential will vanish as t^{-2}. When isotropisation arises such as $\Omega_m \to 0$, we recover the results of case 2A (including the scalar fields asymptotic behaviours) but necessary reality conditions for isotropisation to E_1 and E_2 equilibrium points are cast into respectively $\ell_{\psi_1}^2 < 3/2\gamma$ and $1-\gamma/2 < 2\ell_{\psi_2}(\ell_{\psi_1+2\ell_{\psi_2}})^{-1} < 1$.

Once again when $\Omega_m \to const \neq 0$, we recover the same behaviour for the metric functions as in the presence of a single scalar field despite the unusual form of ω. To check the above limits and inequalities, again we need to know the scalar fields asymptotic behaviours. We have got:

Asymptotic behaviours of the scalar fields for case 2B ($k \neq 0$)
The asymptotic behaviours of the two scalar fields when an isotropic state is reached with $\Omega_m \to const \neq 0$ are those of the two functions ϕ and ψ as $\Omega \to -\infty$ defined by:

$$\dot{\psi} = 3\gamma \frac{U}{U_\psi} \tag{5.53}$$

$$\dot{\phi} = \frac{12\phi}{\sqrt{3+2\omega}} e^{3\left[(1-\gamma/2)-\gamma\int \ell_{\psi 2}\ell_{\psi 1}^{-1} d\Omega\right]} \tag{5.54}$$

5.6 Applications

To illustrate our calculation, we look for isotropisation conditions of some important theories extensively studied in the literature. Remember that we have assumed a positive potential, the respect of the weak energy condition and $\gamma \in [1,2]$. Then, in the following applications, when we will write that isotropisation is impossible, we must keep in mind that it could be wrong if one of the above assumptions were violated. These applications will be illustrated with numerical simulations done with a Runge-Kutta algorithm (order 5) implemented in java. Java program and its user manual may be downloaded at http://luth2.obspm.fr/˜etu/fay/stephane.html. It allows to integrate any hyperextended scalar tensor theories (with varying G, ω and U) with a perfect fluid for any class A Bianchi models written with the Lagrangian or Hamiltonian(with the variables of this work) field equations. The equations system (5.24-5.27) is also implemented and any new equations system or numerical methods may be easily added and should work with all the codes already written if user manual recommendations are followed.

5.6.1 Hybrid inflation

In the introduction we have connected the presence of two scalar fields with higher order theories or hybrid inflation. Hybrid inflation is studied in [114] with a scalar tensor theory defined by:

$$(3+2\omega)\phi^{-2} = 2 \tag{5.55}$$

$$(3+2\mu)\psi^{-2} = 2 \tag{5.56}$$

$$U = 1/4\lambda(\psi^2 - M^2) + 1/2m^2\phi^2 + 1/2\lambda'\phi^2\psi^2 \tag{5.57}$$

m, M, λ and λ' being some constants. It thus corresponds to cases $1A$ and $1B$ defined above. The same type of theory is also used in [115] for similar reasons and from the point of view of topological defects. The potential (5.57) has the symmetry $\phi \leftrightarrow -\phi$ and $\psi \leftrightarrow -\psi$ and is the most general form of a renormalisable potential with this property, apart from the absence of a $\lambda''\phi^4$ term. For a flat FLRW model, inflation stops when the true vacuum state, which corresponds to a global minimum for the potential with $(\phi,\psi) = (0,M)$, is reached. When no perfect fluid is present, we calculate that ℓ_{ϕ_1} and ℓ_{ψ_1} are respectively proportional to $\dot\phi$ and $\dot\psi$ and write:

$$\ell_{\phi_1} = \frac{2\sqrt{2}\phi(m^2 + \lambda'\psi^2)}{\lambda(M^2 - \psi^2)^2 + 2\phi^2(m^2 + \lambda'\psi^2)} \tag{5.58}$$

$$\ell_{\psi_1} = \frac{2\sqrt{2}\psi\left[\lambda'\phi^2 + \lambda(\psi^2 - M^2)\right]}{\lambda(M^2 - \psi^2)^2 + 2\phi^2(m^2 + \lambda'\psi^2)} \tag{5.59}$$

Obviously, with $(\phi,\psi) = (0,M)$, we have $\phi \to 0$ and $M^2 - \psi^2 \to 0$. Then, if we assume that the vanishing of ϕ is slower, faster or of the same order as $M^2 - \psi^2$, we respectively find that ℓ_{ϕ_1}, ℓ_{ψ_1} or the couple $(\ell_{\phi_1},\ell_{\psi_1})$ diverge. Then it is the same for the derivatives of the scalar fields. The first graph of figure 5.11 represents a numerical integration of the scalar fields and illustrates this fact. Consequently, the couple $(\phi,\psi) = (0,M)$ represents an asymptotic state of true vacuum which can not occur with isotropisation of the Bianchi type I model. Moreover, numerical simulations show that the scalar fields are not defined as $\Omega \to -\infty$, thus confirming that this theory can not lead to isotropisation of the Universe. Such a result is interesting because early time inflation is often used to solve some problems of the standard big bang model such as the flatness problem or the isotropy of the cosmological microwave background. However we see that starting from an anisotropic model, the hybrid inflation for the theory defined by (5.55-5.57) is not able to isotropise the Universe.

When a perfect fluid is present, numerical simulations (second and third graphs of figure 5.11) indicate that ϕ would oscillate to 0 whereas ψ would tend to a constant M_0 different from M as $\Omega \to -\infty$. Thus, the potential tends to a constant and not to $V^{-\gamma}$. Consequently isotropisation does not occur when $k \neq 0$. Since it can not arise either when no perfect fluid is present, we conclude that, even when $k \to 0$, isotropisation does not take place at late time.

Hence class 1 isotropisation seems impossible for the theory of this section. Numerical simulations for the system (5.24-5.27) confirm the result and do not show class 2 or 3 isotropisation either.

5.6.2 High-order theories and compactification

Another theory can be defined by the same forms of Brans-Dicke coupling functions but with another form of potential:

$$U = U_0 e^{-\sqrt{2/3}k\phi} e^{-5\sqrt{3}/6k\psi}(e^{\sqrt{3}/2\psi} - 1)^m \tag{5.60}$$

with $k > 0$ and $m > 0$[2]. Such potentials appear when we compactify the space-time and cast high-order theories of gravity into relativistic forms. Hence in [113], conformal transformations are applied to the theory defined by $S = \int d^5x\sqrt{G_5}(\frac{M_5^3}{16\pi}R_5 + \alpha M_5^{-3}R_5^4)$ and lead to the scalar tensor theory defined above with $m = 4/3$, whereas if we consider the action $S = \int d^5x\sqrt{G_5}(\frac{M_5^3}{16\pi}R_5 + bM_5R_5^2 + cM_5^{-3}R_5^4)$, we get a scalar tensor theory with $m = 2$. These actions are related to M-theory compactification. When no perfect fluid is present, using (5.45) and (5.46), we find that near isotropy:

$$\phi \to -\sqrt{2/3}k\Omega + \phi_0 \tag{5.61}$$

2. These assumptions allow to simplify the study.

$$-\sqrt{2/3}k\Omega+\phi_0 \quad\rightarrow\quad -\frac{2\sqrt{2}}{5(5k-3m)}\{2\sqrt{3}m\ln\left[e^{\sqrt{3}\psi/2}(5k-3m)-5k\right] +(5k-3m)\psi\} \tag{5.62}$$

Since we consider $k>0$, ψ does not diverge to $-\infty$ otherwise the left member of equation (5.62) would be complex. Numerical simulations show that ψ tends to $+\infty$ when $\Omega\rightarrow-\infty$ and then, we deduce from (5.62) that $\psi\rightarrow-(5k-3m)(2\sqrt{3})^{-1}\Omega$. This limit will arise in $\Omega\rightarrow-\infty$ if in the same time $5k-3m>0$. An illustration of the two scalar fields asymptotical behaviours has been plotted on the fourth graph of figure 5.11. We calculate that the quantities ℓ_{ϕ_1} and ℓ_{ψ_1} respectively tend to the constants $-k/\sqrt{3}$ and $(3m-5k)/(2\sqrt{6})$. The necessary condition for isotropisation is thus $(11k^2-10km+3m^2)/8<3$. Assuming that $(k,m)\neq(0,0)$, the late time attractor of the metric functions is a power law of the proper time $t^{24\left[8k^2+(5k-3m)^2\right]^{-1}}$. Hence, after some conformal transformations, these theories derived from particle physics can lead to isotropisation of Bianchi type I model as illustrated by figure 5.6.

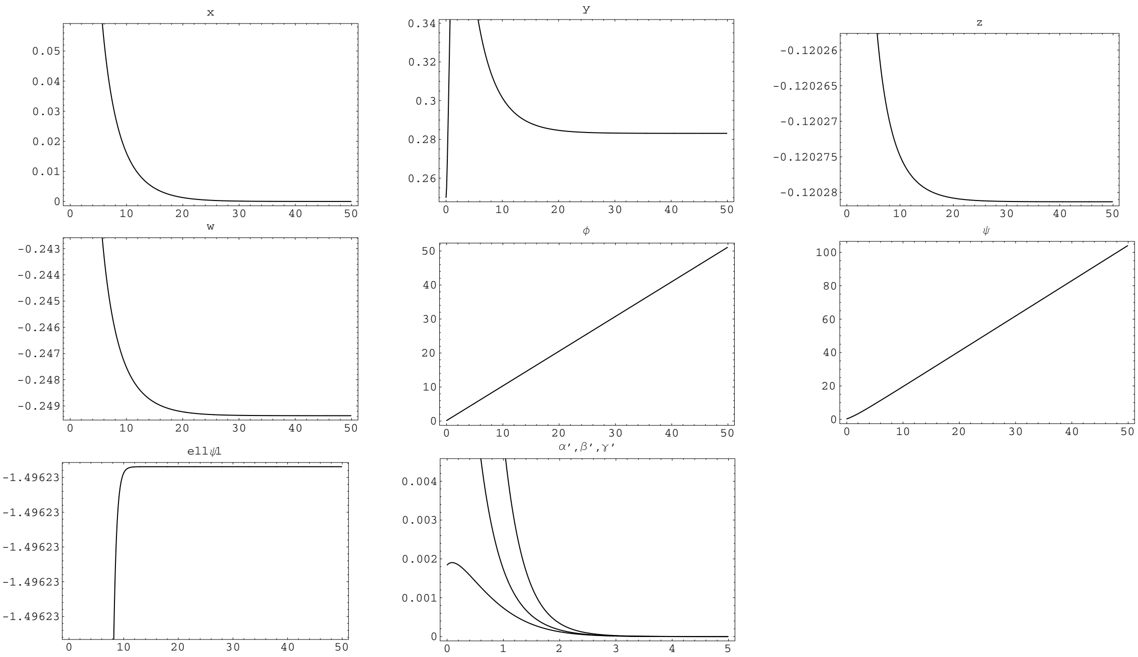

FIG. 5.6 – *These figures, with $-\Omega$ in abscissa, represent successively the behaviours of $(x,y,z,w,\phi,\psi,\ell_{\psi_1})$ for initial condition $(x,y,z,w,\phi,\psi)=(-0.49,0.25,-0.12,-0.15,0.14,0.23)$ and parameters $(U_0,k,m)=(3.2,1.25,-0.36)$ and a dust fluid. Note that ℓ_{ϕ_1} is a constant $-k/\sqrt{3}=-0.721688$. The last figure shows the vanishing of α, β and γ derivatives with respect to the proper time as it should be in case of metric functions convergence to a power law of time. If we take $m=-2.36$, $(11k^2-10km+3m^2)/8>3$ and class 1 isotropisation does not occurs since x tends to a non vanishing constant*

When a perfect fluid is present, numerical analysis of (5.52) shows that scalar fields are defined when $\Omega\rightarrow-\infty$ and ψ may diverge. From the forms of $\dot{\phi}$ and $\dot{\psi}$, it is easy to see that ψ can not tend to $-\infty$ for positive k when $\Omega\rightarrow-\infty$. When $\psi\rightarrow+\infty$, it comes $\ell^2_{\phi_1}+\ell^2_{\psi_1}\rightarrow(11k^2-10km+3m^2)/8$ and thus this theory may isotropise to an equilibrium state whose nature depends on the value of this constant with respect to $3/2\gamma$. This case is illustrated on figure 5.7 where a numerical integration has been performed with $(11k^2-10km+3m^2)/8>3/2\gamma$. Numerical integration for the scalar fields theoretical asymptotical behaviours has also been plotted on the fifth graph of figure 5.11. It also produces some solutions for which ψ vanishes and ϕ tends to a non vanishing constant but then, $\ell^2_{\phi_1}+\ell^2_{\psi_1}$ diverges and class 1 isotropisation should not occur.

5.6.3 A common quadratic potential for a complex scalar field

The theories corresponding to cases $2A$ and $2B$ are related to the presence of complex scalar fields whose Lagrangian is most of times written as[118, 119, 120]:

$$L=R+g^{\mu\nu}\zeta^*_{,\mu}\zeta_{,\nu}-V(|\zeta|^2)+L_m \tag{5.63}$$

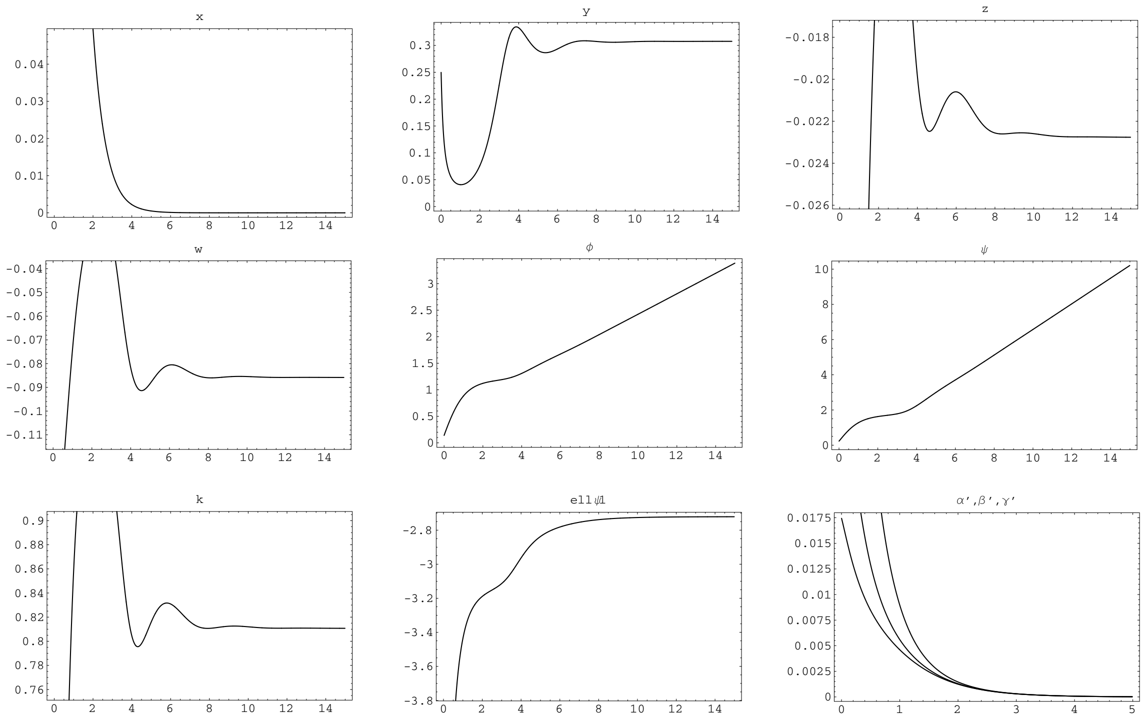

FIG. 5.7 – *These figures, with $-\Omega$ in abscissa, represent successively the behaviours of $(x,y,z,w,\phi,\psi,k,\ell_{\psi_1})$ for initial condition $(x,y,z,w,\phi,\psi) = (-0.49, 0.25, -0.12, -0.15, 0.14, 0.23)$ and parameters $(U_0,k,m) = (3.2, 1.25, -2.36)$ and a dust fluid. Note that ℓ_{ϕ_1} is a constant $-k/\sqrt{3} = -0.721688$. As previously, last figure shows the vanishing of α, β and γ derivatives with respect to the proper time.*

By redefining the scalar field ζ as $\zeta = \psi(\sqrt{2}m)e^{-im\phi}$, it becomes:

$$L = R + 1/2g^{\mu\nu}(\psi^2\phi_{,\mu}\phi_{,\nu} + m^{-2}\psi_{,\mu}\psi_{,\nu}) - U(\psi^2) + L_m \tag{5.64}$$

which corresponds to $3/2+\mu = 1/2m^{-2}\psi^2$ and $3/2+\omega = 1/2\phi^2\psi^2$. Since the potential depends on ψ^2, its most simple and maybe natural form seems to be $U = \zeta\zeta^* = \psi^2$. It is often used in the literature for instance for scalar fields quantization in [118] or to study the genericity of inflation for spatially closed FLRW models in [120]. If we suppose that there is no perfect fluid, then, for E_1 equilibrium point, we get $\psi \to \pm 2m\sqrt{2(\Omega-\psi_0)}$: it is complex when $\Omega \to -\infty$ whereas, by definition, it should be real. For E_2 equilibrium point, we get $\psi \to \psi_0 e^{3/2\Omega}$ whereas now ϕ tends to a complex value instead of a real one. Consequently, for the theory defined by (5.64) with $U = \psi^2$, class 1 isotropisation does not occur at late times. However, numerical simulations of equations (5.24-5.25) reveal that Universe may ("may" and not "must" since $x \to 0$ is a necessary but not sufficient condition for isotropisation.) undergoes a class 3 isotropisation as shows on figure 5.8 with the characteristics explained at the beginning of this work.
If now we assume that a perfect fluid is present, $\psi \to e^{3/2\gamma\Omega}$ and ℓ_{ψ_1} diverges as $e^{-3/2\gamma\Omega}$: then class 1 isotropisation is not possible if $k \neq 0$. However, once again a class 3 isotropisation is possible with k oscillating to a constant as plotted on figure 5.9. If $k \to 0$, as shown above, class 1 isotropisation is impossible but not class 3.

5.6.4 Topological defects

Another type of potential has been used in [121] to study the formation of topological defects after early time inflation. Its form is $U = \lambda/2(\psi^2-\eta^2)^2$, i.e. the so-called wine bottle potential, with λ and η some constants. If we assume that there is no perfect fluid, we calculate for E_1 equilibrium point that $\psi^2 \to -\eta^2 ProductLog(-\eta^{-2}e^{-16m^2\eta^{-2}(\Omega-\phi_0)})$, ϕ_0 being an integration constant[3]. But this last quantity is negative when $\Omega \to -\infty$ and then, again, ψ is asymptotically complex, which does not fit with its definition as a real scalar field. For point E_2, we also find that ψ is complex as $\Omega \to -\infty$ but if the integration constant is complex too. So for both E_1 and E_2 points, an isotropic equilibrium state of class 1 type can not be reached because at least one of the scalar fields is complex at late time.
If now we assume that a perfect fluid is present, we have $\psi^2 \to e^{3/2\gamma(\Omega-\Omega_0)} + \eta^2$, with Ω_0 an integration

3. $ProducLog(z)$ gives the principal solution for w in $z = we^w$.

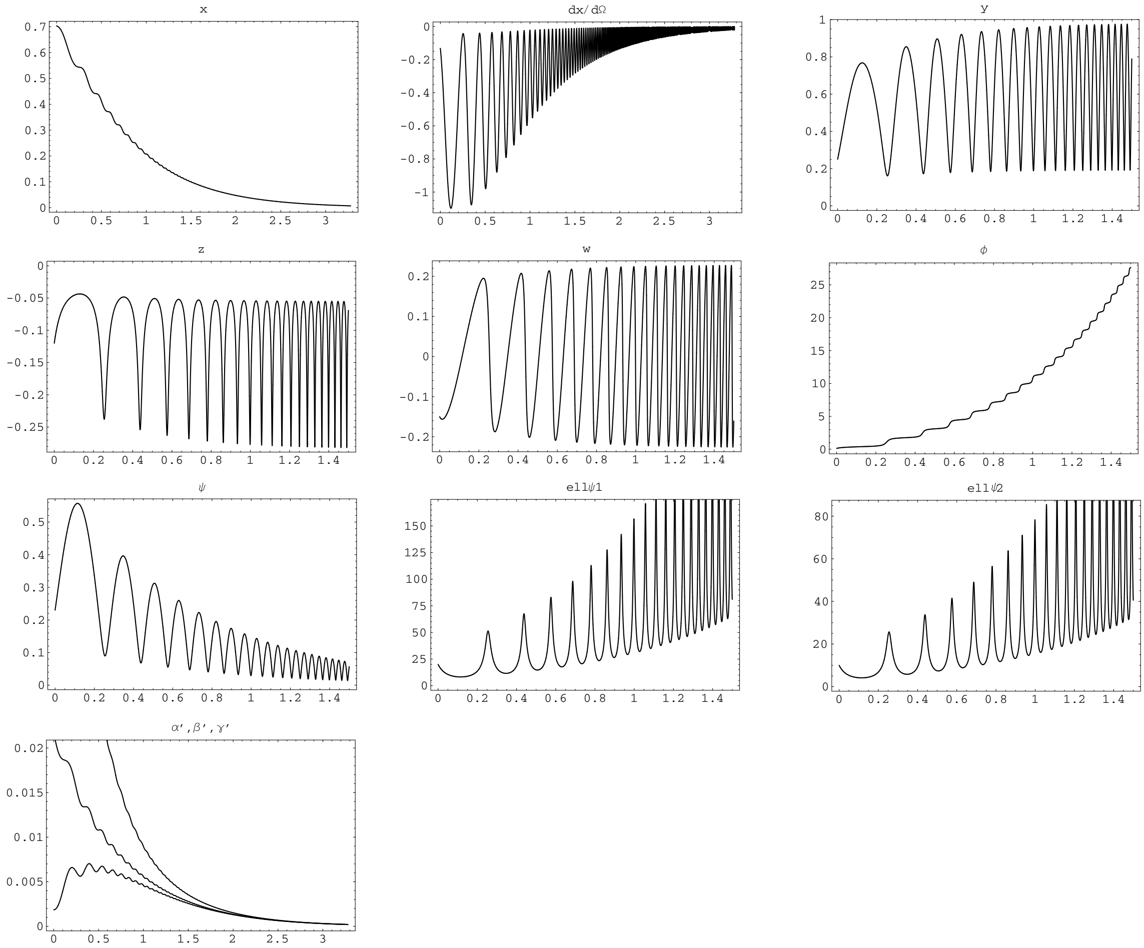

FIG. 5.8 – *These figures, with $-\Omega$ in abscissa, represent successively the behaviours of $(x,\dot{x},y,z,w,\phi,\psi,\ell_{\phi_2},\ell_{\psi_2})$ for initial condition $(x,y,z,w,\phi,\psi) = (-0.70,0.25,-0.12,-0.15,0.14,0.23)$ and parameters $m = -2.3$. x is the only variable to reach equilibrium whereas y, z and w oscillates more and more as $-\Omega$ increases. The scalar fields undergo damped oscillations whereas the oscillations for ℓ_{ϕ_2} and ℓ_{ψ_2} increase. The last figure shows the vanishing of α, β and γ derivatives with respect to the proper time as needed for isotropisation. Note that they oscillate.*

constant. Hence, ℓ_{ψ_1} diverges and class 1 isotropisation does not occur for the same reasons as in the previous application.
However, once again, we have observed class 3 isotropisation with and without matter. In the first case, k tends to a constant with damped oscillations and we have observed that x but also z and the scalar fields could reach equilibrium. This is depicted on figure 5.10. The same remarks apply to the second case. Overall the behaviours of the functions are the same as these shown on figures 5.8.

5.6.5 Bose-Einstein condensate

In [122], Bose-Einstein condensate is studied [4] with a potential of the form $\alpha\psi^2+\beta\psi^4$. Again, assuming no perfect fluid, ψ is complex for E_1 equilibrium point.
Indeed, $\psi \to \left[\alpha(2\beta^{-1})\right]^{1/2}(ProductLog(\alpha^{-1}e^{1+32m^2\beta\alpha^{-1}(\Omega-\psi_0)})-1)^{1/2}$ with ψ_0 an integration constant. Thus, when $\Omega \to -\infty$, the second square root is real only if $\alpha\beta^{-1} < 0$ but then the first one is complex. For E_2 equilibrium point, ψ^2 tends to the constant $-\alpha\beta^{-1}$ with $\alpha < 0$ and $\beta > 0$. In the same time, $\phi \to -2(-3\beta\alpha^{-1})^{1/2}\Omega+\phi_0$, ϕ_0 being an integration constant. Calculating ℓ_{ψ_1} and ℓ_{ψ_2} we get respectively that ℓ_{ψ_1} diverges and $\ell_{\psi_2} \to \pm m\sqrt{-\beta\alpha^{-1}}$. Hence, $2\ell_{\psi_2}(\ell_{\psi_1}+2\ell_{\psi_2})^{-1} \to 0$ and $y \to 0$. We could have a class 2 isotropisation although numerical simulations have failed to confirm it.
If now we consider a perfect fluid, we find $\psi^2 \to -\alpha(2\beta)^{-1} \pm (2\beta)^{-1}(\alpha^2+4\beta e^{-3\gamma(\Omega_0-\Omega)})^{1/2}$, Ω_0 being an integration constant. Then, ℓ_{ψ_1} diverges and an isotropic stable state may be reached only if $k \to 0$. From what we have found above, isotropisation could only occur for E_2 point. However, since the vanishing of k

4. The Lagrangian is different from (5.63).

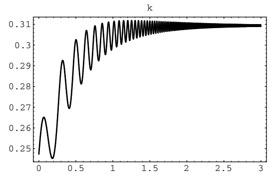

FIG. 5.9 – *If we take into account a perfect fluid, k may reach a constant value during isotropisation.*

needs $1 - \gamma/2 < 2\ell_{\psi_2}(\ell_{\psi_1+2\ell_{\psi_2}})^{-1}$ and the right member of this inequality is vanishing, we conclude that the theory should not undergo class 1 isotropisation.
Once again numerical simulations show class 3 isotropisation, with or without a perfect fluid, and with the same behaviours as those shown on figures 5.8.

We observe that all the theories dealing with complex scalar fields seem to reach isotropisation via class 3 mainly, whereas the others may reach it via class 1.

5.7 Conclusion

We have studied the isotropisation of the flat homogeneous Bianchi type I model filled with a perfect fluid and two real scalar fields. This is an important issue because, as explained in section 5.6, such theories are used to describe hybrid inflation, compactification mechanisms, topological defects or Bose-Einstein condensate which may be related to primordial Universe. Taking the point of view that early Universe is anisotropic, the Lagrangian describing these theories have to be constrained to explain why isotropy arises and what looks like the Universe isotropic state.
To reach this goal, we have made the following **assumptions**:

- We consider the scalar fields either such as $\omega(\phi)$, $\mu(\psi)$ and $U(\phi,\psi)$ or $\omega(\phi,\psi)$, $\mu(\psi)$ and $U(\psi)$ since we thus recover a large number of theories with two real scalar fields or one complex scalar field studied in the literature.
- The weak energy condition is satisfied.
- The potential, which may be considered as a variable cosmological constant, is positive.
- Asymptotically the density parameter of the scalar field should tend to a non vanishing constant value and the ratio of its pressure and energy density to a negative value in accordance with WMAP data.
- The isotropic state is reached sufficiently fast.

We have then found some **necessary conditions** for the Universe isotropisation under the form of some limits and inequalies expressing with respect to the functions ℓ_{ϕ_1}, ℓ_{ϕ_2}, ℓ_{ψ_1} and ℓ_{ψ_2} of the scalar fields ϕ and ψ. The natural outcome of the **Universe isotropisation** may be described as

- A De sitter Universe with a non vanishing cosmological constant
- An Einstein - De Sitter Universe ($e^{-\Omega} \rightarrow t^{\frac{2}{3\gamma}}$) with a vanishing cosmological constant($U \rightarrow t^{-2}$) and a non vanishing perfect fluid density parameter Ω_m.
- A power law expanding Universe ($e^{-\Omega} \rightarrow t^m$, with m the limit of a determined function of the scalar fields) with a vanishing cosmological constant (t^{-2}) and a vanishing perfect fluid density parameter Ω_m.

Note that the potential always tends to a constant or decreases as t^{-2}, whatever the forms of ω, μ and U. In this last case, if the Universe is 15 Gys old, the **cosmological constant** should be $4.96.10^{-57}cm^{-2}$ in agreement with supernovae observations.

When there is no perfect fluid and $\omega(\phi)$, $\mu(\psi)$ and $U(\phi,\psi)$, the results generalise those of [105], where a single scalar field is considered, for any number of minimally coupled scalar fields ϕ_i whose associated Brans-Dicke coupling functions ω_i only depend on ϕ_i: a necessary condition for isotropisation is that $\sum_i \ell_{\phi_i}^2$ tends to a constant smaller than 3. If the constant is vanishing, the Universe tends to a De Sitter model otherwise the metric functions increase as $t^{1/\sum_i \ell_{\phi_i}^2}$. When $\omega(\phi,\psi)$, $\mu(\psi)$ and $U(\psi)$, the results are different because now the factor in front of the ϕ field kinetic term contains the ψ field. Hence, we find two equilibrium points and the power laws representing the asymptotic behaviours of the metric functions when isotropisation occurs are different from the previous case or what we had found in [105].
Considering a perfect fluid modifies the necessary conditions for isotropy even when its density parameter Ω_m tends to vanish. However, in this last case, the asymptotical behaviours of the metric functions and

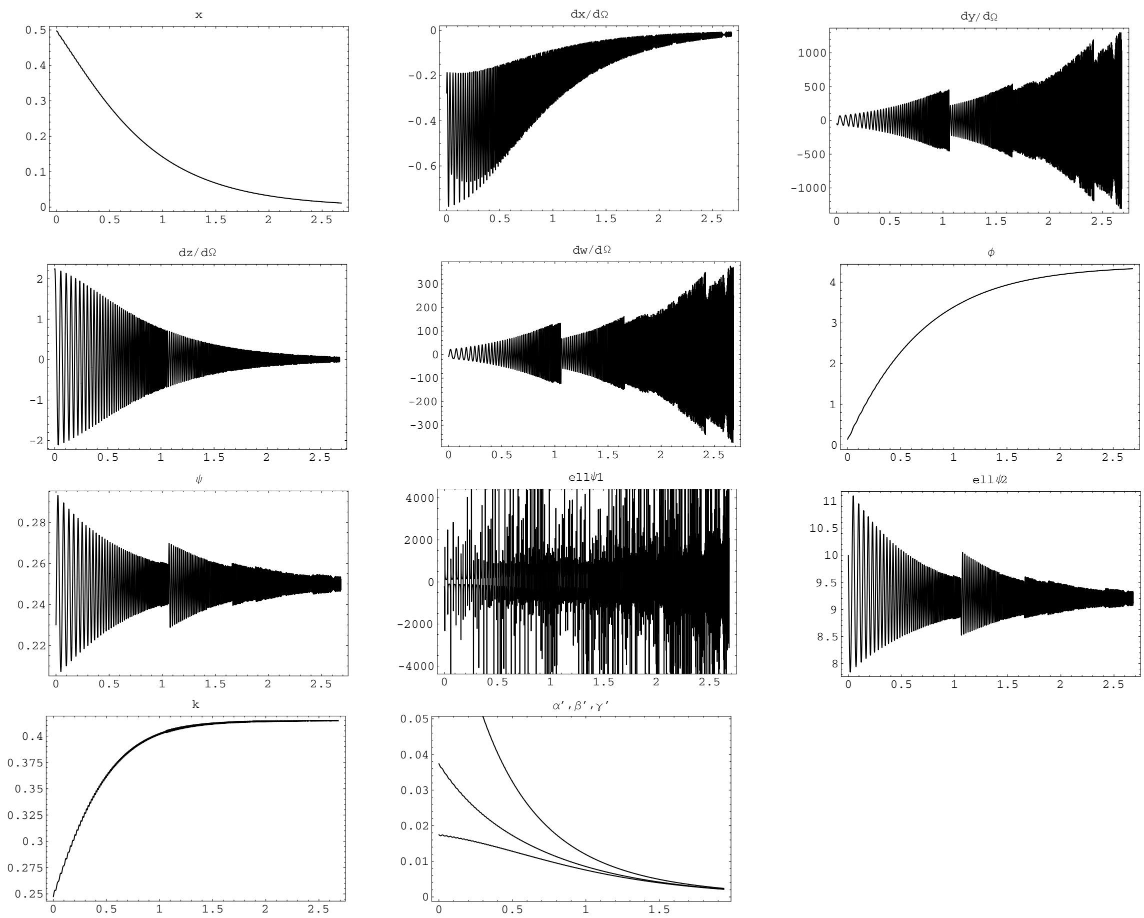

FIG. 5.10 – *These figures, with $-\Omega$ in abscissa, represent successively the behaviours of $(x,\dot{x},\dot{y},\dot{z},\dot{w},\phi,\psi,\ell_{\phi_2},\ell_{\psi_2})$ for initial condition $(x,y,z,w,\phi,\psi)=(0.49,0.25,-0.12,-0.15,0.14,0.23)$ and parameters $(\lambda,\eta)=(0.25,0.25)$. x, z and the scalar fields reach equilibrium whereas ℓ_{ψ_1} undergoes non damped oscillations. The last figure shows the vanishing of α, β and γ derivatives with respect to the proper time.*

potential are the same as without it whereas if Ω_m tends to a nonvanishing constant, the metric functions behave as if they were no scalar field at all, i.e as $t^{\frac{2}{3\gamma}}$. Hence, when Ω_ϕ and Ω_m are asymptotically of the same order, the expansion is decelerated thus preventing to solve the **coincidence problem**[163].

From an observational point of view, this paper shows that the presence of several minimally coupled scalar fields would not be detectable by dynamical observations of a nearly isotropic Universe since the dynamical behaviours of the metric functions or potential are of the same nature as in the presence of a single one, thus showing a **degeneracy problem**.

We have applied our results to several theories and shown, considering the above assumptions, that the model of hybrid inflation considered in [114] does not lead the Universe to an isotropic state on the contrary to some theories coming from compactification process and studied in [113]. All the theories with a complex scalar field and related to scalar fields quantization[118], topological defects[121] or Bose-Eisntein condensate[122] do not undergo a class 1 isotropisation but a class 3, showing among others strongly oscillating behaviours of one or two scalar fields or even of the perfect fluid density parameter.

Acknowledgment

Parts of the calculus and phase portrait diagrams have been made with help of the marvellous DynPack 10.69 package for Mathematica 4 written by Alfred Clark (http://www.me.rochester.edu/courses/ME406/webdown/down.html for download).

FIG. 5.11 – *Some scalar fields numerical integrations for the following applications of section 4: hybrid inflation (*$m=1$, $M=1$, $\lambda=1$, $\lambda'=5$, $\phi(1)=1$, $\psi(1)=1$*) without and with a perfect fluid, high-order theories (*$m=1$, $k=2$, $\phi(1)=-2.5$, $\psi(1)=0.70$*) without and with a perfect fluid.*

5.8 Appendix 1: Generalisation of case $1A$ for n scalar fields

If we consider the presence of n scalar fields ϕ_n and we use the following variables:

$$x = H^{-1} \tag{5.65}$$

$$y = \sqrt{e^{-6\Omega}U}H^{-1} \tag{5.66}$$

$$z_i = p_{\phi_i}\phi_i(3+2\omega_i)^{-1/2}H^{-1} \tag{5.67}$$

i varying from 1 to n, we get the following first order equations system from the Hamiltonian equations:

$$\dot{x} = 3R^2y^2x \tag{5.68}$$

$$\dot{y} = 3y(2\sum_i^n \ell_i z_i + R^2y^2 - 1) \tag{5.69}$$

$$\dot{z}_i = y^2R^2(3z_i - 1/2\ell_i) + 12\sum_{j\neq i}^n \ell_{ij}z_j^2 - 12\sum_{j\neq i}^n \ell_{ji}z_jz_i \tag{5.70}$$

with $\ell_i = \phi_i U_{\phi_i}U^{-1}(3+2\omega_j)^{-1/2}$ and $\ell_{ij} = \phi_i\omega_{j\phi_i}(3+2\omega_i)^{-1/2}(3+2\omega_j)^{-1}$. If we assume that each Brans-Dicke coupling function ω_i only depends on the scalar field ϕ_i as for the case 1, $\ell_{ij}=0$ when $i\neq j$

and the equilibrium points are $(x,y,z_1,..,z_n) = (0,\pm(3-\sum_{i=1}^n \ell_i^2)^{1/2}(\sqrt{3}R)^{-1},1/6\ell_1,..,1/6\ell_n)$. It is thus possible to generalise the results of case 1 by replacing $\ell_{\phi_1}^2+\ell_{\psi_1}^2$ by $\sum_{i=1}^n \ell_i^2$.

5.9 Appendix 2: With a perfect fluid

Equilibrium points calculus when $\ell_{\phi_2}=\ell_{\psi_2}=0$

The equilibrium points are defined by the following (x,y,z,w) values:

- $E_1 = (0,0,0,0)$
- $E_{2,3} = (0,\pm[9(2-\gamma)\gamma-\ell_{\phi 1}^4+12(\gamma-1)\ell_{\psi 1}^2-4\ell_{\psi 1}^4+4\ell_{\phi 1}^2(3\gamma-3-2\ell_{\psi 1}^2)]^{1/2}[\ell_{\phi 1}^2+\ell_{\psi 1}^2]^{-1/2}(2\sqrt{3}R)^{-1},\ell_{\phi 1}(6-3\gamma+2\ell_{\phi 1}^2+2\ell_{\psi 1}^2)\left[12(\ell_{\phi 1}^2+\ell_{\psi 1}^2)\right]^{-1}),$
  $\ell_{\psi 1}(6-3\gamma+2\ell_{\phi 1}^2+2\ell_{\psi 1}^2)\left[12(\ell_{\phi 1}^2+\ell_{\psi 1}^2)\right]^{-1})$
- $E_{4,5} = (0,\pm 1/2\sqrt{3}R^{-1}\left[\gamma(2-\gamma)(\ell_{\phi 1}^2+\ell_{\psi 1}^2)^{-1}\right]^{1/2},1/4\gamma\ell_{\phi 1}(\ell_{\phi 1}^2+\ell_{\psi 1}^2)^{-1},$
  $1/4\gamma\ell_{\psi 1}(\ell_{\phi 1}^2+\ell_{\psi 1}^2)^{-1}$

E_1 may correspond to a class 2 isotropisation. For the other equilibrium points, the constraint implies $k^2 \to 1-\frac{3\gamma}{2(\ell_{\phi 1}^2+\ell_{\psi 1}^2)}$ that is real and non vanishing if $\ell_{\phi 1}^2+\ell_{\psi 1}^2 > 3/2\gamma$. $E_{2,3}$ points will be real only if $\ell_{\phi 1}^2+\ell_{\psi 1}^2 \in [3/2(\gamma-2),3/2\gamma]$ but this condition is incompatible with above constraint on k. Consequently they are eliminated from further considerations. $E_{4,5}$ points are real since $\gamma \in [1,2[$ and thus will be the only one we will consider.

Equilibrium points calculus when $\ell_{\phi_1}=\ell_{\phi_2}=0$

We find 11 equilibrium points that we introduce in the constraint equation to determine the form for k. They can be divided into 3 groups:

- First group:
  The constraint requires $k^2 \to 1$ and then equilibrium points are given by:
  - $E_1 = (0,0,0,0)$
  - $E_{6,7} = (0,0,\pm 1/8\left[-(\gamma-2)^2\ell_{\psi 2}^{-2}\right]^{1/2},1/8(2-\gamma)\ell_{\psi 2}^{-2})$

  The E_1 point is similar to that of the previous section and we make the same remarks. Equilibrium points $E_{6,7}$ are complex and thus eliminated from further considerations.
- Second group:
  The constraint requires $k^2 \to 1-3/2\gamma\ell_{\psi 1}^{-2}$. Introducing this value in the equilibrium points, we get:
  - $E_{2,3} = (0,\pm 1/2R^{-1}\ell_{\psi 1}^{-1}\sqrt{3\gamma(2-\gamma)},0,1/4\gamma\ell_{\psi 1}^{-1})$.
  - $E_{4,5} = (0,\pm(2\sqrt{3}R\ell_{\psi 1})^{-1}\left[9\gamma(2-\gamma)+12(\gamma-1)\ell_{\psi 1}^2-4\ell_{\psi 1}^4\right]^{1/2},0,$
    $(12\ell_{\psi 1})^{-1}(6-3\gamma+2\ell_{\psi 1}^2))$

  $E_{2,3}$ points are real for the considered values of γ. $E_{4,5}$ points are real if $3/2\gamma \in \left[\ell_{\psi 1}^2,\ell_{\psi 1}^2+3\right]$ which is not compatible with a real k arising for $3/2\gamma < \ell_{\psi 1}^2$. Hence, $E_{4,5}$ points are eliminated.
- Third group
  In this last group, the constraint requires $k=0$ or $k\to 0$. Then equilibrium points values and x asymptotic behaviour are the same as in section 5.4.1, although isotropisation conditions are modified as shown in subsection 5.4.2.

It follows that only $E_{2,3}$ equilibrium points may represent an isotropic stable state when $\ell_{\psi 1}^2 > 3/2\gamma$ and k is strictly or asymptotically different from zero.

Chapitre 6

Isotropisation of flat homogeneous Bianchi type I model with a non minimally coupled and massive scalar field

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Abstract

In previous works, we studied the isotropisation of Bianchi class A models with a minimally coupled scalar field. In this paper, we want to extend these results to the case of a non minimally coupled one. To that end, we will first study a scalar tensor theory with a scalar field minimally coupled to the curvature but non minimally coupled to the perfect fluid. Then, we will use a conformal transformation of the metric to generalise our results to a scalar field non minimally coupled to the curvature, i.e. the so called hyperextended scalar tensor theory. Some applications will be made with the Brans-Dicke and low energy string theories.

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6.1 Introduction

There are numerous reasons to consider the presence of some scalar fields in our Universe. Historically, the most famous scalar tensor theory is the Brans-Dicke one which aimed to satisfy Mach ideas. Since the eighties some new justifications have appeared mainly related to particle physics theories. As instance, the supersymmetry assumption supposes the equality between fermionic and bosonic degrees of freedom and needs several scalar fields to exist. Higgs physics also rest upon the presence of such fields. Although these theories are speculative, we hope they will be tested in a near future at LHC, the CERN hadrons collider[190]. Some other reasons to believe that scalar fields exist are related to the cosmology: they could explain the flattening of the spiral galaxies rotation curves[146, 183](dark matter), the late time Universe accelerating expansion[9, 10](dark energy) or the inflation.
The scalar field ϕ we are going to consider is characterised by a Brans-Dicke coupling function ω between the field and the metric, a potential U which describes its self coupling and a coupling function λ between ϕ and a perfect fluid. Generally, it is assumed that $\lambda = 1$, i.e. no coupling between the matter and the scalar. When λ varies, the matter does not follow the spacetime geodesics. The reasons for which we wish to consider such a λ is that from the results we will get about the Universe dynamics for the theory thus defined, we will be able via a conformal transformation, to derive similar results for the Hyperextended Scalar Tensor theory[35] with a perfect fluid, i.e. a theory with a varying gravitation function depending on the scalar field. Such a derivative would be impossible if λ was a constant. We will hence extend our

previous results[105, 109] found with $\lambda = 1$ to the case of a varying gravitation function.
As a geometrical framework, we have chosen to study the homogeneous and spatially flat Bianchi type I model. Anisotropic models allow to generalise the FLRW ones whose high symmetry seems unnatural and to understand why and how isotropy could appear. If the isotropy and homogeneity of our Universe until the decoupling period is well established from the CMB[182], one has to remember that it is always a hypothesis concerning the early Universe. Another justification for this generalisation is that the behaviour of the FLRW model near the singularity is not generic. A generic approach to singularity could be oscillating as the one of the Bianchi type IX model. It has been conjectured by Belinskij, Khalatnikov and Lifchitz (BKL)[184, 185] that it should be shared by the most general anisotropic and inhomogeneous models, conjecture recently revisited by Uggla and others[186]. Here, we will study the Bianchi type I model whose singularity is not oscillatory but which is a spatially flat model in agreement with WMAP data.
Our goal will be to study the isotropisation process of the Bianchi type I model when we consider the presence of a non minimally coupled and massive scalar field. To this end, we will use the ADM Hamiltonian formalism[78, 79] allowing to write the field equations as a first order differential system. Then, we will use the dynamical systems methods[25] to find its isotropic equilibrium states. The plan of the paper is as follows. In section 6.2, we write the Hamiltonian field equations. In section 6.3, some necessary conditions for isotropisation and the behaviours of the metric and potential when such a state is reached are described. In the last section, we summarize our results and study the isotropisation of the Brans-Dicke and low energy string theories when the potential is a power or an exponential law of the scalar field.

6.2 Field equations

The action we described in the introduction writes as:

$$S = (16\pi)^{-1} \int \left[R - (3/2 + \omega)\phi^{,\mu}\phi_{,\mu}\phi^{-2} - U\right] \sqrt{-g}d^4x + S_m(g_{ij},\phi) \tag{6.1}$$

ϕ is the scalar field, ω and U are respectively the Brans-Dicke coupling function and the potential, both depending on ϕ. S_m is the action standing for a perfect fluid coupled to the scalar field with an equation of state $p_m = (\gamma - 1)\rho_m$ and $\gamma \in [1{,}2]$. The metric for the Bianchi type I model is:

$$ds^2 = -dt^2 + R_0^2 g_{ij}\omega^i\omega^j \tag{6.2}$$

The g_{ij} are the metric functions and ω_i the 1-forms specifying the Bianchi type I model. In order to use the Hamiltonian formalism, we have to rewrite this metric following a 3+1 decomposition of spacetime:

$$ds^2 = -(N^2 - N_iN^i)d\Omega^2 + 2N_i d\Omega\omega^i + R_0^2 e^{-2\Omega + 2\beta_{ij}}\omega^i\omega^j \tag{6.3}$$

N and N_i are the lapse and shift functions whereas Ω, a monotonic function of the proper time as we will show it latter, describes the isotropic part of the metric and will be considered as a time coordinate. The 3-volume V of the Universe is thus defined as $V = e^{-3\Omega}$. The β_{ij} stand for the anisotropic part of the metric and have been parameterised by Misner[79] in the following way:

$$\beta_{ij} = diag(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+) \tag{6.4}$$

$$p^i_k = 2\pi\pi^i_k - 2/3\pi\delta^i_k\pi^l_l \tag{6.5}$$

$$6p_{ij} = diag(p_+ + \sqrt{3}p_-, p_+ - \sqrt{3}p_-, -2p_+) \tag{6.6}$$

The p_{ij} are the β_{ij} conjugate momenta. To find the ADM Hamiltonian, we rewrite the action as:

$$S = (16\pi)^{-1} \int (\pi^{ij}\frac{\partial g_{ij}}{\partial t} + \pi^\phi\frac{\partial\phi}{\partial t} + \pi^\psi\frac{\partial\psi}{\partial t} - NC^0 - N_iC^i)d^4x \tag{6.7}$$

where π_ϕ is the scalar field conjugate momentum and N and N_i may be seen as Lagrange multipliers. By varying the action with respect to these quantities, we find the constraints $C_0 = 0$ and $C_i = 0$ with:

$$\begin{aligned} C^0 &= -\sqrt{^{(3)}g}^{(3)}R - \frac{1}{\sqrt{^{(3)}g}}(\frac{1}{2}(\pi^k_k)^2 - \pi^{ij}\pi_{ij}) + \frac{1}{2\sqrt{^{(3)}g}}\frac{\pi_\phi^2\phi^2}{3+2\omega} + \\ &\quad \sqrt{^{(3)}g}U + \frac{1}{\sqrt{^{(3)}g}}\frac{\delta\lambda e^{3(\gamma-2)\Omega}}{24\pi^2} \\ C^i &= \pi^{ij}_{|j} \end{aligned} \qquad \begin{aligned}(6.8)\\(6.9)\end{aligned}$$

where δ and λ are respectively a positive constant and a scalar field function describing the coupling between the scalar field and the perfect fluid. When the action (6.1) is derived using a conformal transformation of a non minimally coupled scalar tensor theory with a gravitation function $G(\phi)$ as described in the appendice, we have the relation $\lambda \propto G^{3(4-3\gamma)}$. Moreover the energy conservation law of the perfect fluid writes $\rho_m = \lambda V^{-\gamma}$. Hence, we will assume that λ is a positive function of ϕ. The constraint $C_i = 0$ is identically satisfied whereas the constraint $C_0 = 0$ gives the ADM Hamiltonian:

$$H^2 = p_+^2 + p_-^2 + 12\frac{p_\phi^2\phi^2}{3+2\omega} + 24\pi^2 R_0^6 e^{-6\Omega}U + \delta\lambda e^{3(\gamma-2)\Omega} \tag{6.10}$$

This Hamiltonian generalises the one got when the scalar field is not coupled to the perfect fluid, i.e. $\lambda = const$. For more details on Hamiltonian formalism, see [125]. The Hamiltonian equations then write:

$$\dot{\beta}_\pm = \frac{\partial H}{\partial p_\pm} = \frac{p_\pm}{H} \tag{6.11}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{12\phi^2 p_\phi}{(3+2\omega)H} \tag{6.12}$$

$$\dot{p}_\pm = -\frac{\partial H}{\partial \beta_\pm} = 0 \tag{6.13}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = -12\frac{\phi p_\phi^2}{(3+2\omega)H} + 12\frac{\omega_\phi \phi^2 p_\phi^2}{(3+2\omega)^2 H} - 12\pi^2 R_0^6 \frac{e^{-6\Omega}U_\phi}{H} - \frac{\delta\lambda_\phi e^{3(\gamma-2)\Omega}}{2H} \tag{6.14}$$

$$\dot{H} = \frac{dH}{d\Omega} = \frac{\partial H}{\partial \Omega} = -72\pi^2 R_0^6 \frac{e^{-6\Omega}U}{H} + 3/2\delta\lambda(\gamma-2)\frac{e^{3(\gamma-2)\Omega}}{H} \tag{6.15}$$

A dot means a derivative with respect to Ω and a subscript ϕ a derivative with respect to the scalar field. We want to rewrite these equations with some bounded variables. For this we define:

$$x = H^{-1} \tag{6.16}$$

$$y = e^{-3\Omega}\sqrt{U}H^{-1} \tag{6.17}$$

$$z = p_\phi \phi (3+2\omega)^{-1/2} H^{-1} \tag{6.18}$$

These new variables will be real if $U > 0$ and $3 + 2\omega > 0$ and thus the weak equivalence principle will be respected. Each of them have a physical interpretation:

- x^2 is proportional to the shear parameter Σ defined in [25].
- y^2 is proportional to $(\rho_\phi - p_\phi)/(d\Omega/dt)^2$, $(d\Omega/dt)^2$ being the Hubble constant when the Universe is isotropic, ρ_ϕ and p_ϕ the density and pressure of the scalar field.
- z^2 is proportional to $(\rho_\phi + p_\phi)/(d\Omega/dt)^2$.
- From the two last points it comes that the density parameter of the scalar field, $\Omega_\phi \propto \rho_\phi/(d\Omega/dt)^2$, is a linear combination of y^2 and z^2 or, when the scalar field is quintessent, that these two variables are proportional to Ω_ϕ.

From the equation (6.10) we get:

$$p^2x^2 + R^2y^2 + 12z^2 + k^2 = 1 \tag{6.19}$$

where we put to simplify

$$k^2 = \delta\lambda e^{3(\gamma-2)\Omega}H^{-2}$$

and the constants $p^2 = p_+^2 + p_-^2$ and $R^2 = 24\pi^2 R_0^6$. The equation (6.19) may be considered as a constraint equation and show that the variables x, y, z and k are bounded. k is not a new independent variable but is related to x, y and ϕ, this last variable being able to diverge. It is proportional to the perfect fluid density parameter $\Omega_m \propto \rho_m/(d\Omega/dt)^2$ and it may be rewritten under the following useful forms:

$$k^2 = \delta\lambda x^\gamma y^{2-\gamma} U^{\gamma/2-1} \tag{6.20}$$

$$k^2 = \delta\lambda x^2 e^{3(\gamma-2)\Omega} \tag{6.21}$$

$$k^2 = \delta y^2 U^{-1}\lambda V^{-\gamma} \tag{6.22}$$

Using the variables (6.16-6.18), the field equations become:

$$\dot{x} = 3R^2y^2x - 3/2(\gamma-2)k^2x \tag{6.23}$$

$$\dot{y} = y(6\ell z + 3R^2y^2 - 3) - 3/2(\gamma-2)k^2y \tag{6.24}$$

$$\dot{z} = R^2y^2(3z - \frac{\ell}{2}) - 3/2(\gamma-2)k^2z - 1/2\ell_m k^2 \tag{6.25}$$

where the quantities ℓ and ℓ_m are defined by $\ell = \phi U_\phi U^{-1}(3+2\omega)^{-1/2}$ and $\ell_m = \phi\lambda_\phi\lambda^{-1}(3+2\omega)^{-1/2}$. ℓ and ℓ_m look each other because of the similar roles of U and λ in the Hamiltonian (6.10) which both are multiplied by an exponential of Ω. The equation for ϕ will be written as:

$$\dot{\phi} = 12z\frac{\phi}{(3+2\omega)^{1/2}} \tag{6.26}$$

Summarising, the seven equations of the Hamiltonian system (6.11-6.15) are reduced to a system of four equations (6.23-6.26) describing the evolution of four variables of which three are bounded. It comes owing to the fact that, for the Bianchi type I model, the hamiltonian equations immediately give $p_\pm = const$ implying $\beta_+ \propto \beta_-$. Moreover, we will choose a diagonal form for the metric, i.e. $N_i = 0$, what allows to get $N = 12\pi R_0^3 H^{-1}e^{-3\Omega}$ with $dt = -Nd\Omega$.

6.3 Stable isotropic states

6.3.1 Defining isotropy

Following Collins and Hawking[108], isotropy arises when $\Omega \to -\infty$, i.e. for a forever expanding Universe such as $d\beta_\pm/dt \propto e^{3\Omega} \to 0$. Moreover, defining $\sigma_{ij} = (de^\beta/dt)_{k(i}(e^{-\beta})_{j)k}$ and $\sigma^2 = \sigma_{ij}\sigma_{ij}$, we must have $\frac{\sigma}{d\Omega/dt} \to 0$. This last condition says that the anisotropy measured locally through the constant of Hubble tends to zero and implies that the shear parameter $x \propto \dot{\beta}_\pm = d\beta_\pm/dtdt/d\Omega \to 0$. Consequently, isotropisation arises when x vanishes in $\Omega \to -\infty$ and is thus a stable state arising for a diverging value of t.

We deduce that the Universe can become isotropic in three different ways respectively named class 1, 2 and 3, and described as follows;

- Class 1: in the vicinity of the isotropy, all the variables (x,y,z) reach equilibrium with $y \neq 0$. It is generally possible to determine the asymptotical forms of the metric functions and potential whatever the forms of ω, U and λ.
- Class 2: in the vicinity of the isotropy, all the variables (x,y,z) reach equilibrium with $y = 0$ and thus $k^2 \to 1 - 12z^2$. Until now, we did not succeed in finding the Universe asymptotical state. A numerical example will be shown in the last section.
- Class 3: in the vicinity of the isotropy, only the variable x reaches equilibrium but not necessarily the others variables. If y and z do not reach equilibrium when $\Omega \to -\infty$, since they have to be bounded, they necessarily oscillate without damping and thus their first derivatives oscillate around zero. This phenomenon will arise if ℓ or/and ℓ_m sufficiently oscillate when $\Omega \to -\infty$ such that the signs of $\dot{y}$ or/and $\dot{z}$ change continuously. Of course in this case, once again it seems difficult to determine the Universe asymptotical state (but we hope not impossible). Class 3 isotropisation has been observed numerically for complex scalar fields in [116].

In this paper, we will consider the first class which is the only one allowing a full description of the Universe asymptotic state when it isotropises. If some properties of the two others classes may be determined, they will be considered in future papers.

6.3.2 Assumptions for the stability of our results

The results of the present paper will consist of the determination of the isotropic equilibrium points, some necessary conditions for isotropisation and the asymptotical behaviours of some functions in the neighbourhood of these points. However we will determine these behaviours by neglecting in the vicinity of the equilibrium the variation of on one hand, a function f of the scalar field whose form depends on the asymptotical behaviour of k and, in the other hand, the ones of the variables (y,z,k). In other words, we will assume that all these quantities tend sufficiently fast to their equilibrium values. We had already talked about this problem in [129] and we reproduce below the discussion of this last paper.

The first type of assumption is related to ℓ and ℓ_m. let $f(\ell,\ell_m)$ be a function of the scalar field that we will define below and tending to a constant f_0 in the vicinity of the isotropic state, vanishing or not. We will assume that this function reaches sufficiently quickly its equilibrium value f_0, i.e.

– When f tends to its constant equilibrium value f_0 (vanishing or not) such as $f \rightarrow f_0 + \delta f$, $\int (f_0 + \delta f) d\Omega \rightarrow f_0\Omega + const.$

We will check this assumption each time we will use our results. If it is not true, the asymptotical behaviours for the metric functions (and potential) will be different from the laws we will derive below. This problem could be overcame since our results allow to calculate $\phi(\Omega)$ and thus $f(\Omega)$. Hence, it should be easy to generalise them by keeping the $\int f d\Omega$ term instead of considering that it tends to $f_0\Omega + const$ but then they would not be on a closed form.

The second type of assumption can not be solved so easily. In the same way, the asymptotical behaviours we will determine will be true only if the variables (y,z,k) tend sufficiently fast to their equilibrium values. For that, we have to make the same kind of assumption for (y,z,k) as for f. For partly solve this problem, it would be necessary to consider some small perturbations of these variables in the vicinity of the equilibrium but until now we did not succeed to get any interesting results, even for the empty flat model.

To summarize, the results of this paper related to asymptotical behaviours will be valid for a class 1 isotropisation if the functions f we will define below and the variables (y,z,k) tend sufficiently fast to their equilibrium values or, from a physical point of view, if the Universe tends sufficiently fast to its isotropic state. The assumption on f may be easily solved but the ones on (y,z,k) need a more careful examination.

6.3.3 Asymptotic state when $k \neq 0$

In what follows, we look for the isotropic states such as $k \neq 0$. The case for which $k \rightarrow 0$ will be analysed in the section 6.3.5.

Equilibrium points

First we calculate the equilibrium points of the equations system (6.23-6.25) and we introduce them in the constraint (6.19) to find k. Then, the equilibrium points write:

$$\begin{aligned}
E_0 &= (0,0,\frac{\ell_m}{3(2-\gamma)}) \\
E_1 &= (0,\pm\frac{1}{2\sqrt{3}R(\ell-\ell_m)}[-4\ell^4+8\ell^3\ell_m-4\ell^2(3+\ell_m^2-3\gamma)- \\
&\quad 12\ell\ell_m(\gamma-1)-9\gamma(\gamma-2)]^{1/2},\frac{6\ell+2\ell^3-6\ell_m-2\ell^2\ell_m-3\ell\gamma}{12\ell(\ell-\ell_m)}) \\
E_2 &= (0,\pm\frac{1}{2R(\ell-\ell_m)}\left[4\ell_m(\ell_m-\ell)-3\gamma(\gamma-2)\right]^{1/2},\frac{\gamma}{4(\ell-\ell_m)})
\end{aligned}$$

E_0 point belongs to class 2 and is thus discarded. For the two other points, k expresses as:

$$k^2 = \frac{2\ell(\ell-\ell_m)-3\gamma}{2(\ell-\ell_m)^2},$$

It is a real as long as:

$$\ell(\ell-\ell_m) > \frac{3}{2}\gamma \tag{6.27}$$

This inequality disagrees with a real value for E_1 points which are then discarded from further considerations. Consequently, the only equilibrium points corresponding to an isotropic class 1 stable state are the E_2 ones. A numerical simulation representing one of them is illustrated on figure 6.1. They are real and bounded if respectively:

$$4\ell_m(\ell_m-\ell) > 3\gamma(\gamma-2) \tag{6.28}$$

$$\ell \not\rightarrow \ell_m \tag{6.29}$$

i.e. $U \not\rightarrow \lambda$. The first condition is automatically satisfied when there is no coupling between the matter and the scalar field($\ell_m = 0$). We will show in section 6.3.4 that ℓ and ℓ_m can not diverge but at the same order and that k tends to a non vanishing constant in the vicinity of the E_2 points.

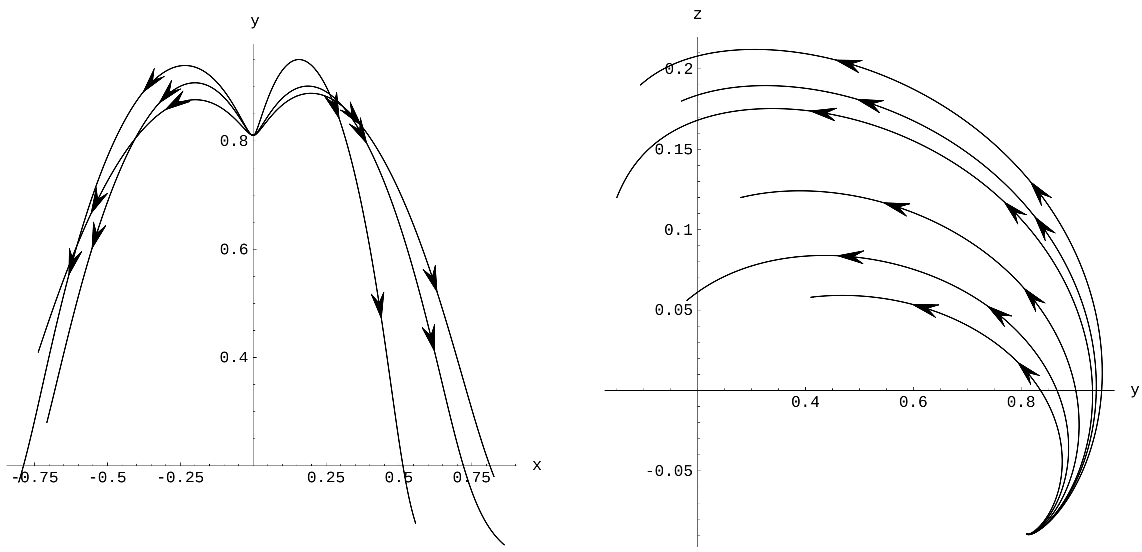

FIG. 6.1 – *Equilibrium point* E_2 *in* $(x,y,z) = (0,0.81,-0.08)$ *when* $k \neq 0$ *and* $(R,\gamma,\ell,\ell_m) = (1,1,-1.23,1.58)$.

Monotonic functions

The equation (6.23) shows that x is a monotonic function of constant sign. We deduce that the metric functions whose derivatives with respect to the proper time write as some linear functions of x can only have a single extremum. From the form of the lapse function N and the relation $dt = -Nd\Omega$, we also derive that Ω is a monotonic function of the proper time whose value in $-\infty$ corresponds to late time epoch when the Hamiltonian is initially positive. These monotonic functions will be also recovered when $k \to 0$

Asymptotical behaviours

The function f we talk about in the subsection 6.3.2 is defined in this subsection as $f = \ell(\ell-\ell_m)^{-1}$. Then linearising the equation (6.23) for x near the equilibrium state, we find that in the vicinity of the isotropy:

$$x \to x_0 e^{-\frac{3[2\ell_m+\ell(\gamma-2)]}{2(\ell-\ell_m)}\Omega},$$

with x_0 an integration constant[1]. x vanishes when $\Omega \to -\infty$ if the reality conditions for k and E_2 are respected. Using the lapse function N and the relation between the proper time t and the time coordinate Ω, i.e. $dt = -Nd\Omega$, we find that $e^{-\Omega}$ tends to the increasing function of the proper time:

$$e^{-\Omega} \to t^{\frac{2(\ell-\ell_m)}{3\ell\gamma}} \tag{6.30}$$

when $\frac{3\ell\gamma}{2(\ell-\ell_m)}$ tends to a non vanishing constant. This is always true since, as it will be shown in the next section, ℓ and ℓ_m can not diverge but at the same order and such that $\ell \not\to \ell_m$. Moreover, the reality condition (6.27) for k shows that ℓ does not vanish. Since in the next section we will also prove that y can not vanish, from y definition we derive that the potential asymptotically disappears as $U \to t^{-2}$.
To check the inequalities (6.27-6.29) when $\omega(\phi)$ and $U(\phi)$ are specified and that are necessary conditions for isotropy, we need to know the asymptotical behaviour of ϕ. For this, we write (6.26) in the neighbourhood of the equilibrium and we get that asymptotically ϕ behaves as the asymptotical solution of the differential equation:

$$\dot{\phi} = 3\gamma(\frac{U_\phi}{U} - \frac{\lambda_\phi}{\lambda})^{-1} \tag{6.31}$$

when $\Omega \to -\infty$. This asymptotical equation is a good illustration of the assumptions we talk about in subsection 6.3.2 for the variables (y,z,k): to determine (6.31), we have replaced z in (6.26) by its equilibrium value $\frac{\gamma}{4(\ell-\ell_m)}$, neglecting any small variation δz when z approaches equilibrium. However, if it tends to $\frac{\gamma}{4(\ell-\ell_m)}$ slower than Ω^{-1}, then δz should be taken into account in (6.31) and not neglected.

1. This result is always valid if ℓ and ℓ_m diverge at the same order.

6.3.4 Some important results for the E_2 points

Integrating (6.31) near equilibrium leads to

$$U \to U_0 \lambda V^{-\gamma}$$

U_0 being an integration constant we then deduce from the expression (6.22) that k tends to a non vanishing constant as long as $y \neq 0$ what is always true. Indeed, the only ways for y to vanish are if $\ell >> \ell_m$ or $4\ell_m(\ell_m - \ell) \to 3\gamma(\gamma - 2)$. In the first case, the constraint or equivalently (6.30) shows that $k \to 1$. But if we consider the form (6.22) for k and the limit $U \to U_0\lambda V^{-\gamma}$, it comes that if $y \to 0 \Rightarrow k \to 0 \neq 1$. For the same reason, y can not tend to 0 when $4\ell_m(\ell_m - \ell) \to 3\gamma(\gamma - 2)$. It follows that y is never vanishing in the vicinity of E_2. We get a similar result if we consider the divergence of ℓ_m when $\ell_m >> \ell$. In this case, y tends to a non vanishing constant and k to 0, which is in disagreement with the form (6.22) of k and the limit $U \to U_0\lambda V^{-\gamma}$. On the other hand, if ℓ and ℓ_m diverge at the same order without converging one to the other, y and k tend to some non vanishing constants and the constraint is respected. Since $\lambda \propto UV^\gamma$ and y can not vanish, we deduce from (6.17) that

$$\lambda \to e^{\frac{3\gamma\ell_m\Omega}{\ell-\ell_m}}$$

and from (6.30) that

$$\lambda \to t^{-2\frac{\ell_m}{\ell}}$$

Thus from the reality condition for k^2, it comes $\lambda > t^{-2(1-\frac{3}{2}\frac{\gamma}{\ell^2})}$. In the same time, the energy densities for the perfect fluid and scalar field write respectively $\rho = \lambda V^{-\gamma} \to U$ and $\rho_\phi = \frac{9}{2}\gamma^2(\ell^{-1} - \ell_m^{-1})^{-2}t^{-2} + \frac{1}{2}U$. When ℓ and ℓ_m tend to some constants, since $U \to t^{-2}$, $p_\phi \propto \rho_\phi \propto \rho_m$: the scalar field and the perfect fluid energy densities behave in the same way. If both ℓ and ℓ_m diverge at the same order, the kinetic term in ρ_ϕ is larger than t^{-2}, $\rho_\phi >> \rho_m$ and the energy density of the scalar field dominates.

6.3.5 Equilibrium point when $k \to 0$

This section is divided in three parts depending on $\ell_m = 0$ strictly, $\ell_m k^2 \to 0$ or $\ell_m k^2 \not\to 0$.

$\underline{\ell_m = 0}$

In this first part, we recall and complete the results got in [109] when no coupling exists between the scalar field and the perfect fluid.

In the vacuum, $k = 0$ strictly and the reality condition for the equilibrium points writes as $\ell^2 < 3$. Near isotropy, the metric functions tend to $t^{\ell^{-2}}$ when ℓ tends to a non vanishing constant or to an exponential when ℓ vanishes. When $k \not\to 0$, the reality condition for the equilibrium points is $\ell^2 > 3/2\gamma$ and the metric functions tend to $t^{\frac{2}{3\gamma}}$. When $k \to 0$, we recover the same values for the equilibrium points as when no perfect fluid is present[105] but now, $k \to 0$ implies $\ell^2 < 3/2\gamma$. The asymptotical behaviour of the metric functions is the same as without a perfect fluid. We had not noticed this last inequality in [109] nor that $U \to V^{-\gamma}$ when $k \to const \neq 0$.

Once again, these results have been got by making the assumptions of subsection 6.3.2 with now $f = \ell^2$ when $k = 0$ or $k \to 0$. When they are not true, meaning that the Universe does not reach its isotropic equilibrium state sufficiently quickly, the asymptotical behaviours of U and $e^{-\Omega}$ are generally different. As instance when $k = 0$ strictly and if ℓ^2 vanishes as $n\Omega^{-1}$ with $n < 0$, the integral of f does not tend to a constant and we can show that the potential will diverge as $(-\Omega)^{-2n}$ and the metric functions will tend to $exp\left[\frac{n+1}{12\pi R_0^3 x_0} t^{1/(n+1)}\right]$ with $n \in \,]-1,0[$ such as the Universe be expanding. This solution is different from the classical solutions found when we neglect the variation of ℓ near equilibrium and it shows that the assumptions on f that we will also use in the next sections, have to be checked each time we apply our results to a specific scalar-tensor theory.

$\underline{\ell_m k^2 \to 0}$

If $k \to 0$ such as $\ell_m k^2 \to 0$, again we recover the same equilibrium points and behaviour for x as in the vacuum, i.e. $x \to x_0 e^{(3-\ell^2)\Omega}$ with $\ell^2 < 3$ such that $x \to 0$ in $\Omega \to -\infty$ and the equilibrium points are real. Consequently, using the form (6.21) for k^2, it comes $k^2 \to \lambda e^{2(3/2\gamma - \ell^2)\Omega}$ when $\Omega \to -\infty$. When ℓ^2 tends to a non vanishing (vanishing) constant smaller than 3, $e^{-\Omega} \to t^{\ell^{-2}}$ (respectively $e^{-\Omega} \to e^{(12\pi R_0^3 x_0)^{-1} t}$). Hence, $k \to 0$ if

$$\lambda e^{2(3/2\gamma - \ell^2)\Omega} \to 0 \tag{6.32}$$

and thus $\lambda < t^{-2(1-\frac{3}{2}\frac{\gamma}{\ell^2})}$ (respectively $\lambda < e^{3\gamma(12\pi R_0^3 x_0)^{-1}t}$). In the same way, $\ell_m k^2 \to 0$ if:

$$\ell_m \lambda e^{2(3/2\gamma-\ell^2)\Omega} \to 0$$

Contrary to the case $\ell_m = 0$, the condition $k \to 0$ does not automatically restrict the set of ℓ allowing isotropisation: it is the form of λ which will lay down the law. Since we consider a class 1 isotropisation such as $y \neq 0$ and $k \to 0$ we have $\lambda V^{-\gamma} << U$ and thus $\rho_\phi - p_\phi >> \rho_m$: the scalar field energy density dominates the Universe.
The asymptotical behaviour of the scalar field when $\Omega \to -\infty$ is given by[105]:

$$\dot{\phi} = 2\frac{\phi^2 U_\phi}{U(3+2\omega)} \tag{6.33}$$

$\ell_m k^2 \not\to 0$
Since $k \to 0$ whereas $\ell_m k^2 \not\to 0$, it means that ℓ_m have to diverge. The equilibrium points when $\ell_m k^2 \not\to 0$ write:

$$E_3 = (0, \pm R^{-1}, 0)$$

and are such as $k^2 = -\ell\ell_m^{-1}$. These points are approached in the same way as the one of the figure 6.1. We need $\ell << \ell_m$ and $\ell\ell_m^{-1} < 0$ such as respectively k vanishes and is real. It is also necessary that ℓ tends to a non vanishing constant or diverges with $z\ell$ being bounded, such as $\ell_m k^2$ be non vanishing. Mathematically, the E_3 point could be the asymptotical limit of E_2 when ℓ_m diverges and $\ell << \ell_m$. However, this divergence is forbidden by the constraint.
Near E_3, we find that $x \to e^{3\Omega}$, indicating that the Universe tends to a De Sitter model, i.e. $e^{-\Omega} \to e^{(12\pi R_0^3 x_0)t}$, and the potential to the constant $(Rx_0)^{-2}$. As previously, we have then $k \to 0$ if:

$$\lambda e^{3\gamma\Omega} \to 0$$

i.e. $\lambda < e^{3\gamma(12\pi R_0^3 x_0)^{-1}t}$. In the same way, $\ell_m k^2$ does not vanish if:

$$\ell_m \lambda e^{3\gamma\Omega} \not\to 0$$

Again, y being different from 0 and considering the form (6.22) for k^2, we have $\rho_m = \lambda V^{-\gamma} << U$ such as $k \to 0$ and thus $\rho_m << \rho_\phi - p_\phi$: the Universe is scalar field dominated. From (6.21) and the limit of k near equilibrium, we determine the scalar field asymptotical behaviour:

$$\delta\frac{1}{\lambda_\phi}\frac{U_\phi}{U} = e^{3\gamma\Omega} \tag{6.34}$$

6.4 Discussion

The discussion is divided in three parts. In the first one we summarize our results and in the second one we consider some applications for Brans-Dicke and low energy string theories. We conclude in the third one.

6.4.1 Summary

We have studied the necessary conditions which may lead the Universe to a class 1 isotropisation in three ways depending if k does not vanish, vanishes with $\ell_m k^2 \to 0$ or with $\ell_m k^2 \not\to 0$. We have assumed that $3 + 2\omega$ and U were some positive functions of the scalar field and that the isotropic state was reached sufficiently fastly. Below we summarize our results.

Case 1: $k \not\to 0$
Let us define the quantities $\ell = \phi U_\phi U^{-1}(3+2\omega)^{-1/2}$ and $\ell_m = \phi\lambda_\phi\lambda^{-1}(3+2\omega)^{-1/2}$. Let p_ϕ, ρ_ϕ and ρ_m be respectively the pressure and density of the scalar field, the density of the perfect fluid. Some necessary conditions for Bianchi type I isotropisation in presence of a massive scalar field minimally coupled to the curvature but not minimally coupled to the perfect fluid are:
- $\ell \not\to \ell_m$ *(equilibrium points are bounded)*
- $4\ell_m(\ell_m - \ell) > 3(\gamma-2)\gamma$ *(reality condition)*
- $\ell(\ell - \ell_m) > \frac{3}{2}\gamma$ *(reality condition)*
- ℓ *and* ℓ_m *are bounded or diverge in the same way (the constraint is respected)*

When isotropy is approached, the metric functions behave as $t^{\frac{2(\ell-\ell_m)}{3\ell\gamma}}$, $\lambda \rightarrow t^{-2\frac{\ell_m}{\ell}}$ *whereas the potential decreases as* t^{-2}. *When* ℓ *and* ℓ_m *do not diverge, there is an equilibrium between the scalar field and the perfect fluid:* $\rho_\phi \propto p_\phi \propto \rho_m$. *When both diverge,* $\rho_\phi - p_\phi >> \rho_m$ *and the Universe is scalar field dominated. Asymptotically, the scalar field checks the relation* $U \rightarrow U_0\lambda e^{3\gamma\Omega}$.

This last expression allows to determine the asymptotical form of ϕ and thus these of ℓ and ℓ_m. Note that in the case $\ell_m = 0$[109], the metric functions asymptotical behaviour does not depend on ϕ and is always $t^{\frac{2}{3\gamma}}$, thus forbidding any late time acceleration. Hence, it is the existence of a coupling between the scalar field and the perfect fluid which allows the appearance of an accelerated expansion when $k \rightarrow const \neq 0$. Then, since when ℓ and ℓ_m are bounded we have $\rho_\phi \propto \rho_m$, it follows that $\Omega_\phi \propto \Omega_m$ and the coincidence problem could be solved.

Case 2: $k \rightarrow 0$ *and* $\ell_m k^2 \rightarrow 0$

Let us define the quantities $\ell = \phi U_\phi U^{-1}(3+2\omega)^{-1/2}$ *and* $\ell_m = \phi\lambda_\phi\lambda^{-1}(3+2\omega)^{-1/2}$. *Some necessary conditions for Bianchi type* I *isotropisation in presence of a massive scalar field minimally coupled to the curvature but not minimally coupled to the perfect fluid are:*

- $\ell^2 < 3$ *(reality condition)*
- $\lambda e^{2(3/2\gamma-\ell^2)\Omega} \rightarrow 0$ *(Condition for* $k \rightarrow 0$*)*
- $\ell_m\lambda e^{(3\gamma-2\ell^2)\Omega} \rightarrow 0$ *(Condition for* $\ell_m k^2 \rightarrow 0$*)*

If ℓ^2 *tends to a non vanishing constant, the metric functions tend to* $t^{\ell^{-2}}$ *and the potential vanishes as* t^{-2}. *If* ℓ^2 *vanishes, the Universe tends to a De Sitter model and the potential to a constant. In any cases* $\rho_\phi - p_\phi >> \rho_m$ *and the Universe is scalar field dominated. The asymptotical behaviour for the scalar field is this of the asymptotical solution of* $\dot\phi = 2\frac{\phi^2 U_\phi}{U(3+2\omega)}$.

These results include the ones got in the vacuum[105]. For sake of clarity, we have chosen to express the limits $k \rightarrow 0$ and $\ell_m k \rightarrow 0$ above (as well as below) depending on $e^{-\Omega}$ and ϕ, these two quantities being asymptotically defined with respect to the proper time t by the behaviours of the metric functions and potential.

Case 3: $k \rightarrow 0$ *and* $\ell_m k^2 \not\rightarrow 0$

Let us define the quantities $\ell = \phi U_\phi U^{-1}(3+2\omega)^{-1/2}$ *and* $\ell_m = \phi\lambda_\phi\lambda^{-1}(3+2\omega)^{-1/2}$. *Some necessary conditions for Bianchi type* I *isotropisation in presence of a massive scalar field minimally coupled to the curvature but not minimally coupled to the perfect fluid are:*

- ℓ_m *diverges and* $\ell \rightarrow const \neq 0$ *or diverges such that* $z\ell \rightarrow 0$ *(condition for* $\ell_m k^2 \rightarrow 0$*)*
- $\ell << \ell_m$ *or* $\lambda e^{3\gamma\Omega} \rightarrow 0$ *(condition for* $k \rightarrow 0$*)*
- $\ell\ell_m^{-1} < 0$ *(reality condition)*

The Universe tends to a De Sitter model and the potential to a constant. Since $\rho_\phi - p_\phi >> \rho_m$, *the scalar field asymptotically dominates the Universe and checks the equation* $\delta\frac{1}{\lambda_\phi}\frac{U_\phi}{U} = e^{3\gamma\Omega}$.

The cases with $k \not\rightarrow 0$ and $k \rightarrow 0$ are strictly separated by asymptotical behaviour of λ since the first one implies $\lambda > t^{-2(1-\frac{3}{2}\frac{\gamma}{\ell^2})}$ and the second one $\lambda < t^{-2(1-\frac{3}{2}\frac{\gamma}{\ell^2})}$ (or $\lambda < e^{3\gamma(12\pi R_0^3 x_0)^{-1}t}$ when $\ell \rightarrow 0$). The two cases such as $k \rightarrow 0$ are distinguished by the fact that the first one tends to a De Sitter model when $\ell \rightarrow 0$ and the second one when $\ell \neq 0$.

6.4.2 Applications

In what follows, we are going to use a conformal transformation of the metric described in the appendice and casting the minimally coupled scalar-tensor theory (6.1) in the Einstein frame where our results take place into a non minimally coupled scalar-tensor theory (6.35) in the Brans-Dicke frame. Obviously when isotropy arises in the Einstein frame, it also occurs in the Brans-Dicke frame and thus necessary conditions for isotropy are the same in both frames. However, the metric functions generally behave differently.
We will illustrate each application with some figures showing the behaviours of x, y, z, k, ϕ and ℓ in the Einstein frame and in the Ω time with initial conditions $\phi_0 = 0.14$, $y_0 = 0.25$, $z_0 = 0.12$. x_0 is calculated using the constraint (6.19) with $p_+^2 + p_-^2 = p^2 = 1$, $R = 1$ and $\delta = 1$ (the constant in the definition of k). The behaviours in the Brans-Dicke frame of the metric functions α, β and γ and their derivatives will be also shown but in the proper time t with initial conditions $\alpha_0 = -1.53$, $\beta_0 = -1.25$, $\gamma_0 = 0.12$, $d\alpha_0/d\tau_0 = 2.48$, $d\beta_0/d\tau_0 = 1.55$ and $d\gamma/d\tau_0 = 0.33$, the τ time being defined as $dt = Vd\tau$. In this

aim, we have numerically integrated the Lagrangian field equations and $d\phi_0/d\tau_0$ has been calculated using the constraint of this formalism. Each time, a dust fluid and a null initial time have been considered. These figures have been got using a 5 order Runge-Kutta method implemented in java. Java is an oriented object language and the application we have developed allows to separate the equations to be integrated from the integration method. Hence, one can add easily a new integration method without having to rewrite the equations and vice versa, thus producing easily and quickly numerical integrations [2].

Brans-Dicke theory with an exponential potential
Consider the class of theories defined by (6.1) such that:

$$\begin{aligned}\omega &= \omega_0\\ U &= \phi^{-2}e^{n\phi}\\ \lambda &= \phi^m\end{aligned}$$

Using the conformal transformation, it can be cast into the non minimally coupled scalar field theory (6.35) defined by:

$$\begin{aligned}G &= \phi^{\frac{m}{3(4-3\gamma)}}\\ \omega &= \left[\frac{3}{2}(1-\frac{m^2}{9(4-3\gamma)^2})+\omega_0)\right]\phi^{\frac{-m}{3(4-3\gamma)}-1}\\ U &= \phi^{-2(1+\frac{m}{3(4-3\gamma)})}e^{n\phi}\end{aligned}$$

The Brans-Dicke theory with an exponential potential is then recovered for $m = 3(3\gamma-4)$.
The quantities ℓ and ℓ_m are defined by:

$$\ell = \frac{n\phi - 2}{\sqrt{3+2\omega_0}}$$

$$\ell_m = \frac{m}{\sqrt{3+2\omega_0}}$$

ℓ_m can not diverge and consequently the <u>case 3</u> never happens. Moreover $3+2\omega_0$ have to be positive. For the <u>case 1</u>, near the isotropic equilibrium state, we have for the scalar field:

$$e^{n\phi}\phi^{-(2+m)} \rightarrow U_0 e^{3\gamma\Omega}$$

Since ℓ is bounded, ϕ can not diverge and should asymptotically vanish, implying that $m < -2$ and finally $\phi \rightarrow e^{\frac{3\gamma}{-(2+m)}\Omega}$. The second reality condition implies then:

$$\frac{4(2+m)-3\gamma(3+2\omega_0)}{2(3+2\omega_0)} > 0$$

But since $m < -2$, $\gamma > 0$ and $3+2\omega_0 > 0$, this condition can not be satisfied and consequently, a class 1 isotropisation does not arise.

Let us consider the <u>case 2</u>. Integrating the differential equation for ϕ, we get:

$$\phi \rightarrow \frac{2}{n - \phi_0 e^{\frac{4\Omega}{3+2\omega_0}}}$$

Then, when $\Omega \rightarrow -\infty$, $\phi \rightarrow 2n^{-1}$, $\ell \rightarrow 0$ and λ tends to the constant $(2n^{-1})^m$ implying $n > 0$. If the Universe isotropises, it will tend to a De Sitter one. Remark that ϕ and thus n have to be positive such that λ be a real function.

Using the conformal transformation, when isotropisation occurs in the Brans-Dicke frame where ϕ is non minimally coupled to the curvature and since λ tends to a constant, the metric also tends to a De Sitter one (see figure 6.2).
A class 2 isotropisation is also possible when $n < 0$ and is plotted on figure 6.3. As above noted, such a range of n is impossible for class 1 isotropisation since λ would be a complex function. It is the only

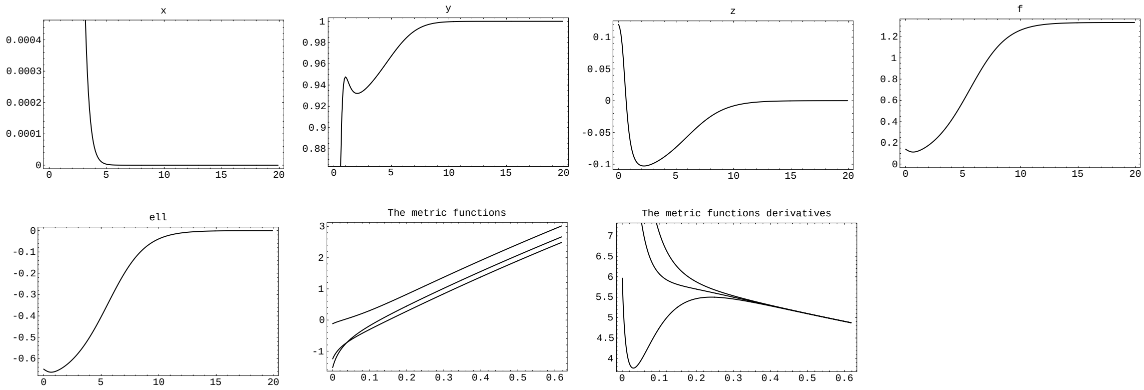

FIG. 6.2 – *These figures represent the approach for class 1 isotropisation when $\omega_0 = 2.3$, $n = 1.5$ and $m = 1.1$. As expected, x tends to 0, ϕ to the constant $2/n = 1.33$ and ℓ (here named ell) to 0. The convergence of ϕ to a constant is in accordance with the fact that U also tends to a constant and the Universe to a De Sitter model. In the Brans-Dicke frame, the derivatives of the metric functions α, β and γ tend to the same behaviour showing isotropisation.*

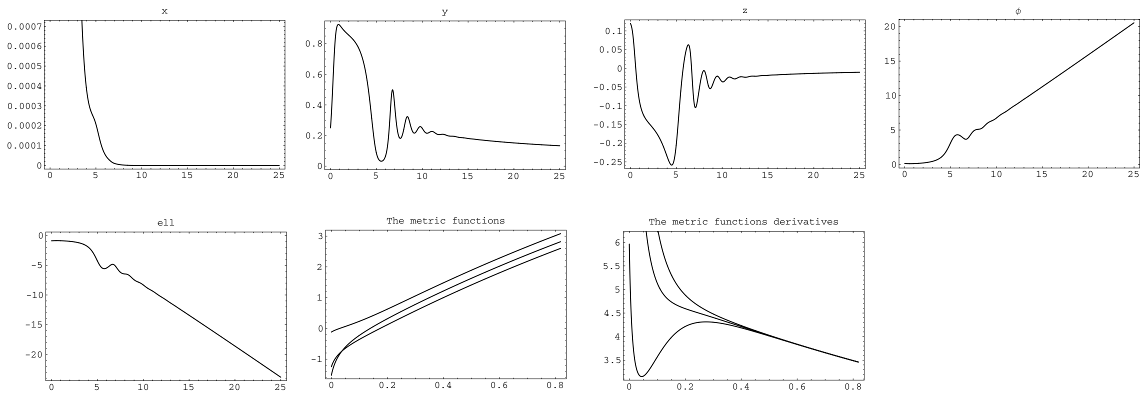

FIG. 6.3 – *These figures represent the approach for class 2 isotropisation when $\omega_0 = 2.3$, $n = -3.1$ and $m = 1.1$. x always tends to 0 but also y. ϕ and thus ℓ diverge. Note that ϕ, y, z and ℓ undergo damped oscillations. In the Brans-Dicke frame, the derivatives of the metric functions tend to the same behaviour showing isotropisation.*

example of class 2 isotropisation we have found until now.

Brans-Dicke theory with a power potential

Consider the class of theories defined by the Lagrangian (6.1) such that:

$$\begin{aligned} \omega &= \omega_0 \\ U &= \phi^n \\ \lambda &= \phi^m \end{aligned}$$

If we apply again the conformal transformation, we obtain the non minimally coupled scalar tensor theory defined by:

$$\begin{aligned} G &= \phi^{\frac{m}{3(4-3\gamma)}} \\ \omega &= \left[\frac{3}{2}\left(1 - \frac{m^2}{9(4-3\gamma)^2}\right) + \omega_0\right]\phi^{\frac{-m}{3(4-3\gamma)}-1} \\ U &= \phi^{n-\frac{2m}{3(4-3\gamma)}} \end{aligned}$$

The Brans-Dicke theory with a power law potential is recovered for $m = 3(3\gamma - 4)$.
We calculate that:

$$\ell = \frac{n}{\sqrt{3 + 2\omega_0}}$$

2. The application is available on request

$$\ell_m = \frac{m}{\sqrt{3+2\omega_0}}$$

with $3+2\omega_0 > 0$. Anew ℓ_m can not diverge and the case 3 is excluded. For the case 1, it is necessary that $n \neq m$ such that $\ell \not\to \ell_m$. Asymptotically the scalar field behaves as:

$$\phi \to \phi_0 e^{-\frac{3\gamma}{m-n}\Omega}$$

Consequently, in $\Omega \to -\infty$, $\phi \to 0$(ϕ diverges) if $m-n<0$ (respectively $m-n>0$). The reality conditions write:

$$4m(m-n)+3\gamma(2-\gamma)(3+2\omega_0) > 0$$
$$2n(n-m)-3\gamma(3+2\omega_0) > 0$$

The second one will be respected if n¿0(n¡0) when $\phi \to 0$(respectively ϕ diverges). We find then that if an isotropic state is reached, the metric functions tend to $t^{\frac{2(n-m)}{3n\gamma}}$ and λ to $t^{-\frac{2m}{n}}$.

Using the conformal transformation, we deduce for the non minimally coupled theory that the metric functions will tend to:

$$t^{\frac{m(8-5\gamma)+2n(3\gamma-4)}{\gamma[m+3n(3\gamma-4)]}}$$

All these behaviours are illustrated on figure 6.4.

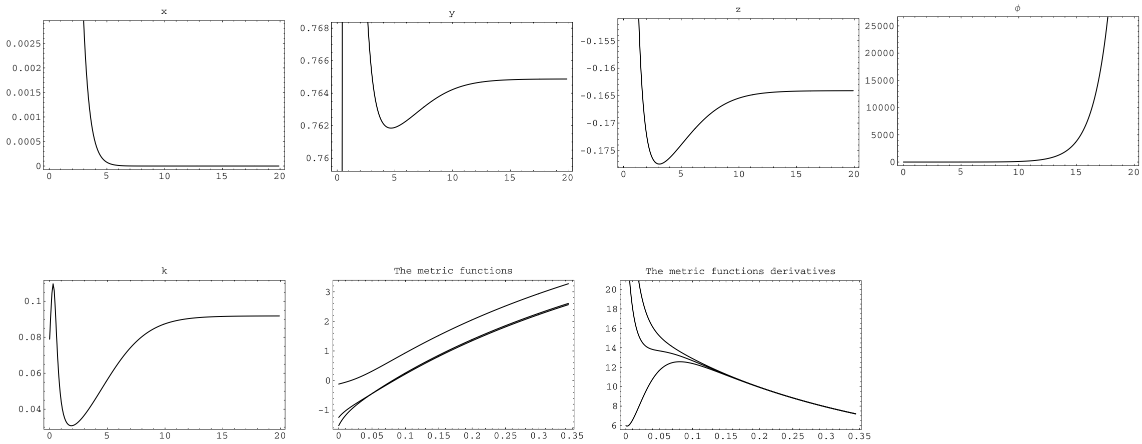

FIG. 6.4 – *These figures represent the approach for class 1 isotropisation when $\omega_0 = 2.3$, $n = -3.1$ and $m = 1.1$. Anew x tends to vanish, y and k to some non vanishing constants showing that $U \propto \lambda e^{3\gamma\Omega}$. ϕ diverges since $m-n>0$. In the Brans-Dicke frame, the Universe isotropises.*

For the case 2, we get for ϕ:

$$\phi \to e^{\frac{2n}{3+2\omega_0}\Omega}$$

Hence k will vanish when $\Omega \to -\infty$ if $2n(m-n)+3\gamma(3+2\omega_0) > 0$ and the reality condition for the equilibrium points will be respected if $n^2(3+2\omega_0)^{-1} < 3$. The metric functions then tend to $t^{(3+2\omega_0)n^{-2}}$ when $n \neq 0$ or to a De Sitter model when $n=0$.

In the Brans-Dicke frame where the scalar field is non minimally coupled to the curvature, the metric function will tend to:

$$t^{\frac{mn+3(3\gamma-4)(3+2\omega_0)}{n[m+3n(3\gamma-4)]}}$$

when $n \neq 0$. If $n=0$, the behaviour of the metric functions is the same as in the Einstein frame and the Universe tends to a De Sitter model. This case is illustrated on figure 6.5

Low energy string theory with an exponential potential

We consider the theory defined by (6.1) and such that:

$$\begin{aligned} \omega &= \omega_0\phi^2+\omega_1 \\ U &= e^{n\phi} \\ \lambda &= e^{m\phi} \end{aligned}$$

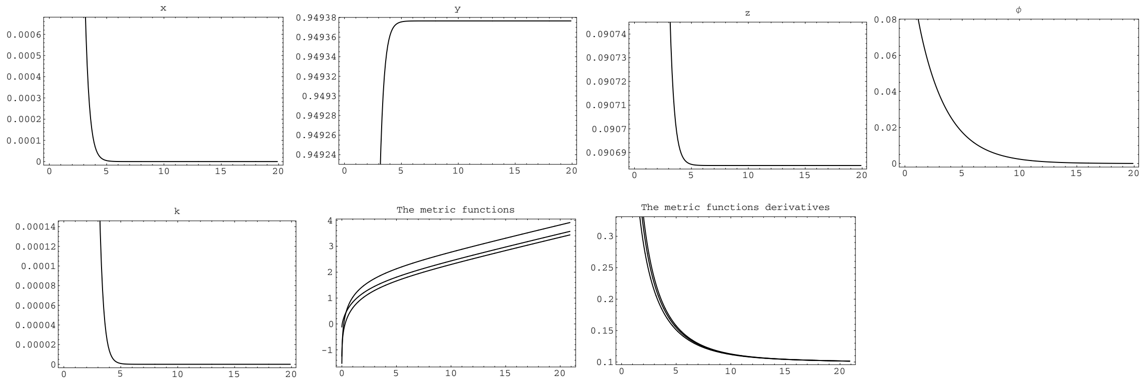

FIG. 6.5 – *These figures represent the approach for class 1 isotropisation when $\omega_0 = 2.3$, $n = 1.5$ and $m = 1.1$. Here, k tends to vanish and the scalar field energy density dominates the Universe.*

Applying the conformal transformation, we define the following non minimally coupled scalar tensor theory:

$$\begin{aligned} G &= e^{\frac{m}{3(4-3\gamma)}\phi} \\ \omega &= \left[\frac{\frac{3}{2}+\omega_0\phi^2+\omega_1}{\phi^2} - \frac{3m^2}{18(4-3\gamma)^2}\right]\phi e^{\frac{-m}{3(4-3\gamma)}\phi} \\ U &= e^{(n-\frac{2m}{3(4-3\gamma)})\phi} \end{aligned}$$

The low energy string theory with an exponential potential is then recovered when $m = 3(4-3\gamma), \omega_0 = 5/2$ and $\omega_1 = -3/2$.
We calculate ℓ and ℓ_m and we obtain:

$$\ell = \frac{n\phi}{\sqrt{3+2\phi^2\omega_0+2\omega_1}}$$

$$\ell_m = \frac{m\phi}{\sqrt{3+2\phi^2\omega_0+2\omega_1}}$$

These expressions show that we will never have $\ell << \ell_m$ and thus the case 3 never occurs. For the case 1, it is necessary that $m \neq n$. Moreover, we find for the scalar field:

$$\phi \to \phi_0 + \frac{3\gamma\Omega}{n-m}$$

Hence, ϕ diverges and ℓ and ℓ_m tend to some constants which will be real if $\omega_0 > 0$. The reality conditions write:

$$2m(m-n) + 3(2-\gamma) > 0$$

$$n(n-m) - 3\gamma\omega_0 > 0$$

ω_0 being positive, the second condition needs $n(n-m) > 0$ and thus $n \neq 0$. Consequently, when isotropisation arises, the metric functions and λ respectively tend to $t^{2\frac{n-m}{3n\gamma}}$ and $t^{-2\frac{m}{n}}$.

We deduce that in the Brans-Dicke frame, when isotropisation arises, the metric functions will tend to:

$$t^{\frac{m(8-5\gamma)+2n(3\gamma-4)}{\gamma[m+3n(3\gamma-4)]}}$$

This case is represented on the figure 6.6.

Concerning the case 2, the scalar field asymptotically behaves as:

$$\phi \to \frac{2n(\Omega-\phi_0) \pm \sqrt{8\omega_0(3+2\omega_1)+4n^2(\phi_0-\Omega)^2}}{4\omega_0}$$

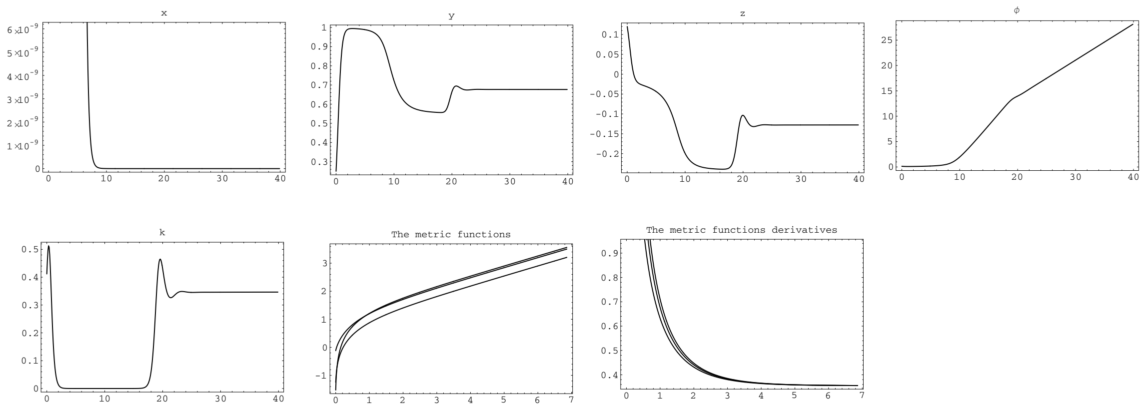

FIG. 6.6 – *These fi gures represent the approach for class 1 isotropisation when $\omega_0 = 2.3$, $n = -3.1$ and $m = 1.1$. k tends to a constant showing the equilibrium between the scalar fi eld and the perfect fluid. Remark the existence before this equilibrium of a period during which the density of the scalar fi eld dominated the one of the perfect fluid.*

Consequently, depending on the sign of the square root, we have two branches such that $\phi \to 0$ or $\phi \to n\omega_0^{-1}\Omega$. For the first one, $\ell \to 0$ and the Universe tends to a De Sitter model. The limit allowing the vanishing of k is always respected. For the second one, $\ell \to n(2\omega_0)^{-1/2}$ and thus, isotropisation needs $\omega_0 > 0$ and $n^2(2\omega_0)^{-1} < 3$. If $n \neq 0$, the metric functions tend to $t^{\frac{2\omega_0}{n^2}}$ and the limit allowing k to vanish is satisfied if $\ell^2 < \frac{3\gamma}{2}$. If $n = 0$, the Universe tends to a De Sitter model and the limit $k \to 0$ is always satisfied.

Again, in the Brans-Dicke frame, we deduce that when isotropisation arises and the scalar field vanishes or $n = 0$, the metric functions tend to the same form as in the Einstein frame because λ tends to a constant. When the scalar field diverges and $n \neq 0$, they tend to:

$$t^{\frac{n^2(9\gamma-13)+3(7\gamma-8)\omega_0}{n^2(9\gamma-13)+3\gamma\omega_0}}$$

Low energy string theory with a power potential

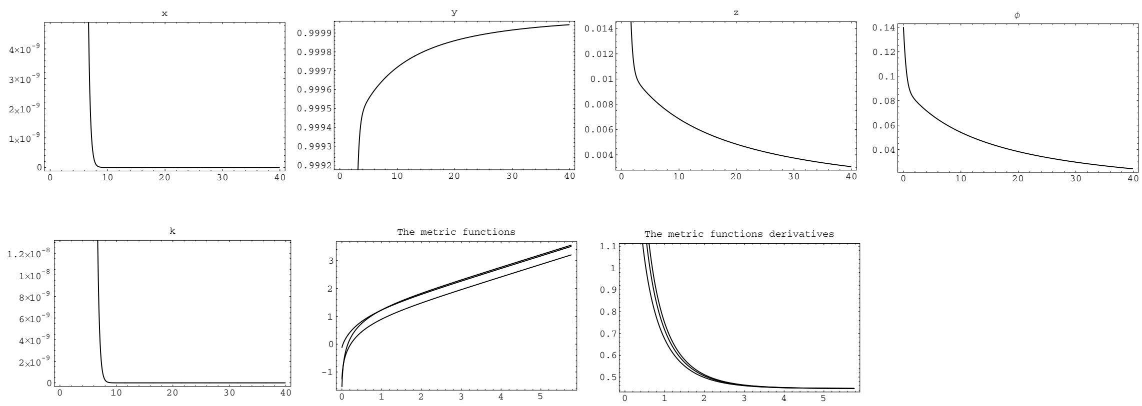

FIG. 6.7 – *These fi gures represent the approach for class 1 isotropisation when $\omega_0 = 2.3$, $\omega_1 = 0.5$, $n = 1.5$ and $m = 1.1$. k tends to vanish showing that the scalar fi eld energy density dominates the one of the perfect fluid. In the same way, ϕ vanishes.*

We now consider the minimally coupled Lagrangian (6.1) with:

$$\begin{aligned} \omega &= \omega_0\phi^2 + \omega_1 \\ U &= \phi^p e^{n\phi} \\ \lambda &= e^{m\phi} \end{aligned}$$

Applying the conformal transformation, it is cast into the following non minimally coupled theory:

$$G = e^{\frac{m}{3(4-3\gamma)}\phi}$$

$$\begin{aligned}
\omega &= \left[\frac{\frac{3}{2}+\omega_0\phi^2+\omega_1}{\phi^2} - \frac{3m^2}{18(4-3\gamma)^2}\right]\phi e^{\frac{-m}{3(4-3\gamma)}\phi} \\
U &= \phi^p e^{(n-\frac{2m}{3(4-3\gamma)})\phi}
\end{aligned}$$

The law energy string theory with a power potential is recovered when $m = 3(4-3\gamma)$, $n = 2$, $\omega_0 = 5/2$ and $\omega_1 = -3/2$.
Calculating ℓ and ℓ_m, we get:

$$\ell = \frac{p+n\phi}{\sqrt{3+2\phi^2\omega_0+2\omega_1}}$$

$$\ell_m = \frac{m\phi}{\sqrt{3+2\phi^2\omega_0+2\omega_1}}$$

Again, it is impossible that ℓ_m diverges and in the same time $\ell << \ell_m$. Thus the case 3 is excluded. For the case 1, we show that the scalar field behaves as:

$$\phi = p(m-n)^{-1}ProductLog((n-m)e^{3\gamma p^{-1}(\Omega-\phi_0)})$$

When $p\gamma^{-1} > 0$, the scalar field vanishes, otherwise it diverges. Then, $(n-m)p^{-1}$ have to be positive otherwise ϕ is complex.
When $\phi \to 0$, it is necessary that $3+2\omega_1 > 0$ such that ℓ and ℓ_m be real and the reality conditions for the equilibrium points reduce to $2p^2 - 3\gamma(3+2\omega_1) > 0$. Then, the metric functions tend to $t^{\frac{2}{3\gamma}}$ and λ to a constant. This case is plotted on figure 6.8.
When $\phi \to \infty$, it is necessary that $\omega_0 > 0$ such as ℓ and ℓ_m be real and $n \neq m$ such as ℓ does not tend to

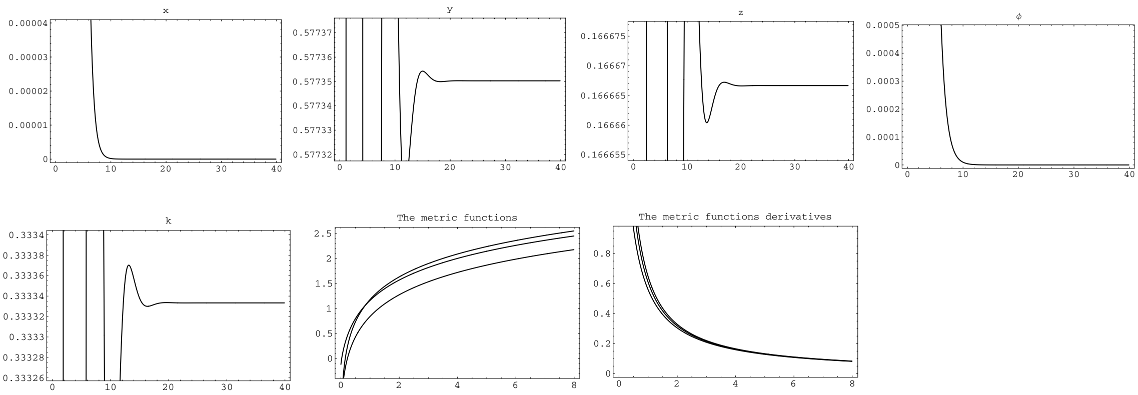

FIG. 6.8 – *These figures represent the approach for class 1 isotropisation when $\omega_0 = 2.3$, $\omega_1 = 0.5$, $n = -3.1$, $m = 1.1$ and $p = 3$. k oscillates to a constant and ϕ vanishes. Note the strong oscillations of y, z and k.*

ℓ_m. The reality conditions for the equilibrium points write then

$$2m(m-n) + 3\gamma(2-\gamma)\omega_0 > 0$$

$$n(n-m) - 3\gamma\omega_0 > 0$$

implying that $n(n-m) > 0$ and $n \neq 0$. The metric functions tend to $t^{\frac{2(n-m)}{3n\gamma}}$ and $\lambda \to t^{-2\frac{m}{n}}$. Some figures similar to the figure 6.8 but with diverging ϕ may be obtained.

In the Brans-Dicke frame, the metric functions tend to the same form as in the Einstein frame during isotropisation if $\phi \to 0$. When ϕ diverges, they tend to:

$$t^{\frac{m(8-5\gamma)+2n(3\gamma-4)}{\gamma[m+3n(3\gamma-4)]}}$$

Let us examine the case 2. The scalar field is such that:

$$\phi_0 + 1/2\left[\frac{(3+2\omega_1)\ln\phi}{p} - \frac{n^2(3+2\omega_1)+2p^2\omega_0}{pn^2}\ln(p+n\phi) + \frac{2\omega_0\phi}{n}\right] \to \Omega$$

Hence, it exists three cases such that $\Omega \to -\infty$.
In the first one, ϕ tends to vanish and it is then necessary that $p > 0$ and $3 + 2\omega_1 > 0$. $\ell \to p(3 + 2\omega_1)^{-1/2}$ and thus we need $p^2(3 + 2\omega_1)^{-1} < 3$. The metric functions tend to $t^{(3+2\omega_1)/p^2}$. k always tends to 0 as long as $\ell^2 < 3/2\gamma$. This case is shown on figure 6.9. Since ϕ vanishes, λ tends to a constant and the results are the same in the Brans-Dicke frame.
In the second one, ϕ diverges as $\frac{n}{2\omega_0}\Omega$. It must be positive and $\omega_0 > 0$ thus implying $\phi \to +\infty$ and $n < 0$.

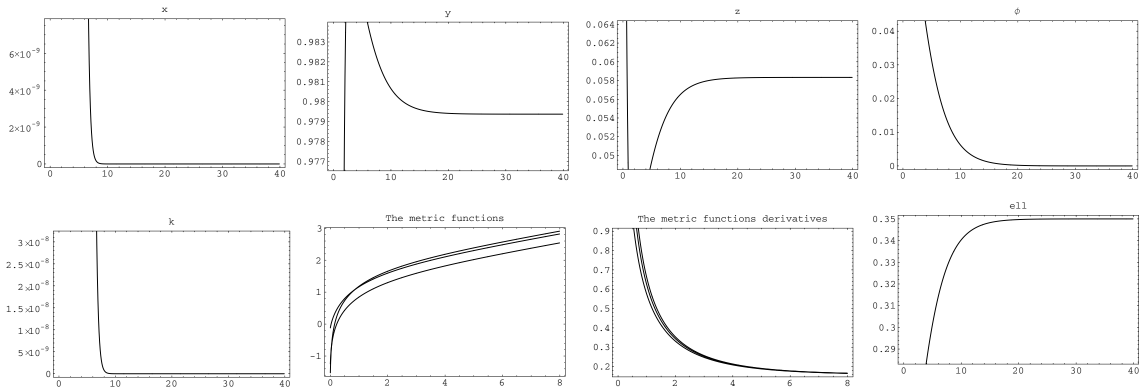

FIG. 6.9 – *These figures represent the approach for class 1 isotropisation when $\omega_0 = 2.3$, $\omega_1 = 0.5$, $n = -3.1$, $m = 1.1$ and $p = 0.7$. k and ϕ vanishes. ℓ tends to 0.35 which is smaller than $3/2\gamma = 3/2$*

Then, ℓ tends to $n(2\omega_0)^{-1/2}$ and it follows that a necessary condition for isotropisation is $n^2(2\omega_0)^{-1} > 3$. Then, the metric functions tend to $t^{\frac{2\omega_0}{n^2}}$ and k to 0 if $n(m - 2n) + 6\gamma\omega_0 > 0$. In the Brans-Dicke frame, we find that the metric functions tend to $t^{\frac{mn+12\omega_0(3\gamma-4)}{mn}}$.
In the third one, $\phi \to -pn^{-1}$ which implies $\left[-n^2(3 + 2\omega_1) - 2k^2\omega_0\right](pn^2)^{-1} > 0$. Then, $\ell \to 0$ and the Universe tends to a De Sitter model. The condition $k \to 0$ is always respected. Once again λ tends to a constant and in the Brans-Dicke frame, the metric functions tend to the same form as in the Einstein frame.

6.4.3 Conclusion

We have found some necessary conditions for isotropisation of Bianchi type I model with a massive scalar field, minimally coupled to the curvature but not to the perfect fluid. They depend on the asymptotical behaviours of k and the product $k\ell_m$. We have then deduced the asymptotical behaviours of the metric functions and the potential in the vicinity of the isotropy. A possible solution to the coincidence problem has also been found. Through some applications, we have shown how to extend our results to a scalar field non minimally coupled to the curvature. The necessary conditions for isotropisation are the same in the Einstein or Brans-Dicke frames but the asymptotical behaviour of the metric functions are different and has to be determined via a conformal transformation. We have thus studied the isotropisation of the Brans-Dicke and low energy string theories with a power or exponential laws of the scalar field for the potential. In a next work, we will examine the quintessential properties of the class 1 for a minimally coupled scalar tensor theory and for the Bianchi class A models.

Parts of the calculus and phase portrait diagrams have been made with help of the marvellous DynPack 10.69 package for Mathematica 4 written by Alfred Clark (http://www.me.rochester.edu/ courses/ME406/webdown/down.html for download).

6.5 Appendix: Perfect fluid conservation law when it is non minimally coupled to the scalar field

In this appendice, we calculate the energy momentum conservation law of the perfect fluid when it is non minimally coupled to the scalar field. This calculus is also made in [123] and more particularly in [124]. Let us consider the Lagrangian of a non minimally coupled scalar field also known as hyperextended scalar tensor theory[35]:

$$L = (G^{-1}R - \omega\phi^{-1}\phi_{,\mu}\phi^{,\mu} - U + T^{\alpha\beta}\delta g_{\alpha\beta})\sqrt{g} \tag{6.35}$$

Then we define a conformal transformation of the metric:

$$g_{\alpha\beta} = G\bar{g}_{\alpha\beta} \tag{6.36}$$

$$dt = \sqrt{G}d\bar{t} \tag{6.37}$$

The frame related to $g_{\alpha\beta}$ is usually called the Brans-Dicke frame whereas the one related to $\bar{g}_{\alpha\beta}$ is called the Einstein frame. In both cases, t and $\bar{t}$ are the proper times such as the 00 metric functions components are -1. Applying the transformation (6.37) casts the Lagrangian (6.35) into:

$$L = \left[\bar{R} - (3/2(G^{-1})^2_\phi G^2 + \omega G\phi^{-1})\phi_{,\mu}\phi^{,\mu} - G^2 U + G^3 T^{\alpha\beta}\delta\bar{g}_{\alpha\beta}\right]\sqrt{\bar{g}} \tag{6.38}$$

where the scalar field is now coupled non minimally with the perfect fluid but minimally with the curvature. Consequently, it comes:

$$\begin{aligned}\bar{T}^{\alpha\beta} &= G^3 T^{\alpha\beta}\\ \bar{T} &= G^2 T\end{aligned}$$

We deduce the following energy conservation law:

$$\begin{aligned}\bar{T}^{\alpha\beta}_{;\alpha} &= 3G_{,\alpha}G^2 T^{\alpha\beta} \text{ (since } T^{\alpha\beta}_{;\alpha} = 0)\\ \bar{T}^{\alpha\beta}_{;\alpha} &= 3G_{,\alpha}G^2 g^{\alpha\beta}T^{\alpha}_{\alpha}\\ \bar{T}^{\alpha\beta}_{;\alpha} &= 3G_{,\alpha}G^2 G^{-1}\bar{g}^{\alpha\beta}G^{-2}\bar{T}\\ \bar{T}^{\alpha\beta}_{;\alpha} &= 3G_{,\alpha}G^{-1}\bar{g}^{\alpha\beta}\bar{T}\\ \bar{T}^{\alpha\beta}_{;\alpha} &= -3\frac{dG}{dt}G^{-1}\bar{T} \text{ (since } G = G(t))\end{aligned}$$

Let us remark that in [124], this law is interpreted as the action of a force on matter due to the variability of the rest masses. Consequently, matter does not follow the spacetime geodesics. To simplify the calculations, we put $p^* = G^2 p$ and $\rho^* = G^2\rho$. Hence, we have $\bar{T}^{\alpha\beta} = (\rho^* + p^*)u^\alpha u^\beta + \bar{g}^{\alpha\beta}p$. Moreover, we have assumed $p = (\gamma - 1)\rho$. Thus, it comes:

$$\begin{aligned}\bar{T}^{0\beta}_{;\beta} &= -3\frac{dG}{dt}G^{-1}(3p^* - \rho^*)\\ \frac{d\rho^*}{dt} + (\rho^* + p^*)V^{-1}\frac{dV}{dt} &= -3\frac{dG}{dt}G^{-1}(3\gamma - 4)\rho^*\\ \rho^{*-1}\frac{d\rho^*}{dt} + \gamma V^{-1}\frac{dV}{dt} &= -3\frac{dG}{dt}G^{-1}(3\gamma - 4)\\ \rho^* V^\gamma = G^{3(4-3\gamma)}\end{aligned}$$

From this last result and the expression for the Lagrangian L_m for a perfect fluid calculated in [125, pages 48-52], we are able to determine the form of H_m, the term describing the matter in the ADM Hamiltonian. Indeed, we have:

$$\begin{aligned}L_m &= T^{\alpha\beta}\delta g_{\alpha\beta}\sqrt{g}\\ &= -8\pi R_0^3 N e^{-3\Omega}\rho\\ &= -8\pi R_0^3 \bar{N} e^{-3\bar{\Omega}}\rho^*\\ &= -8\pi R_0^3 \bar{N} e^{-3\bar{\Omega}} G^{3(4-3\gamma)} V^{-\gamma}\end{aligned}$$

and consequently:

$$H_m = -24\pi^2\bar{g}^{1/2}L_m = 192\pi^3 R_0^3 G^{3(4-3\gamma)} e^{3(\gamma-2)\bar{\Omega}} > 0 \tag{6.39}$$

We will write symbolically this relation under the form $H_m = \delta\lambda(\phi)e^{3(\gamma-2)\bar{\Omega}}$.

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Résumé

Cette thèse étudie les modèles cosmologiques homogènes mais anisotropes en théories tenseur-scalaire. Son but est de déterminer les propriétés que doivent avoir ces théories afin que ces modèles possèdent asymptotiquement les caractéristiques dynamiques de notre Univers actuel ou apportent une réponse à certains de ses problèmes comme ceux de la constante cosmologique. La première partie de la thèse est consacrée à une introduction historique et à une justification physique des théories tenseur-scalaires de la gravitation et des modèles cosmologiques anisotropes. La seconde partie détaille les notions mathématiques nécessaires à la compréhension de cette thèse, à savoir la classification des cosmologies anisotropes et l'écriture des équations de champs dans le formalisme Lagrangien et Hamiltonien. La troisième partie est composée d'une série de sept articles montrant comment l'on peut parvenir à contraindre les théories tenseur-scalaires à l'aide de solutions exactes, en exigeant que l'Univers possède certains comportements dynamiques (expansion, inflation, etc), soit dépourvu de singularité ou possède une symétrie de Noether. Dans la quatrième partie, le processus d'isotropisation des modèles anisotropes est étudié en détail pour de nombreuses classes de théories tenseur-scalaires. Des contraintes nécessaires à l'isotropisation, les comportements asymptotiques des fonctions métriques et du potentiel au voisinage de cet état sont déterminés et le phénomène de quintessence analysé. Un lien entre les champs scalaires quintessents qui pourraient peupler notre Univers et la matière noire dans les galaxies (1 article) est montré. Les six articles à l'origine de ce chapitre sont reproduits dans la sixième partie qui tient lieu d'appendice. Nous concluons dans la cinquième partie.

Abstract

This thesis studies the homogeneous but anisotropic cosmological models in scalar-tensor theories. Its goal is to determine the properties of these theories so that these models asymptotically behave as our current Universe or bring some responses to some of its problems like the cosmological constant or coincidence problems. The first part of the thesis is devoted to a historical introduction and a physical justification of the scalar-tensor theories and anisotropic cosmological models. The second part details the mathematical tools, necessary to understand this thesis, namely anisotropic cosmologies classification and the form of the field equations in the Lagrangian and Hamiltonian formalism. The third part is made up of a series of seven published papers and shows how one can constraint the scalar-tensor theories using exact solutions, requiring some dynamical characteristics for the Universe(expansion, inflation, etc), preventing the singularity occurence or showing Noether symmetries. In the fourth part, the isotropisation process of the anisotropic models is studied in detail for many classes of scalar-tensor theories. Necessary constraints for isotropisation, the asymptotic behaviors of the metric functions and potential in the vicinity of this state are determined and the quintessence phenomenon is analyzed. A link between quintessent scalar fields which could populate our Universe and dark matter in the galaxies is shown. The six papers at the origin of this chapter are reproduced in the sixth part. We conclude in the fifth part.